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ON THE CONVERGENCE OF SEQUENCES OF FIXED POINTS IN PARTIAL METRIC SPACES

Về sự hội tụ của dãy điểm bất động trong không gian mêtric riêng

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ABSTRACT

In this note, we shall give some results on the relationship between the convergence of sequences of mappings and the convergence of their fixed points in partial metric spaces. The results certainly are derived from the metric spaces by Nadler's and some existing results on the topic in the references. We also give some examples to illustrate for the results.

Keywords: *partial metric spaces, fixed point theorems, sequence of contraction generally contractive mappings*

TÓM TẮT

Trong bài báo này, chúng tôi đưa ra một số kết quả về mối quan hệ giữa sự hội tụ của dãy các ánh xạ với dãy điểm bất động của chúng trong không gian mêtric riêng. Các kết quả có nguồn gốc từ những kết quả của Nadler và một số tác giả khác đã xét với trường hợp không gian mêtric. Ngoài ra, chúng tôi cũng đưa ra một số ví dụ minh hoạ cho tính hiệu lực của các kết quả mà chúng tôi đạt được.

Từ khóa: *Dãy điểm bất động, không gian mêtric riêng, dãy các ánh xạ*

1. Introduction

Let (f_n) be a sequence of self-mappings over a metric space (X, d). Suppose that this sequence (f_n) converges to a self-mapping f, defined on a metric space (X, d), in some sense. It is quite natural to ask about the relationship between the convergence of the sequence (f_n) and the convergence of their fixed point. This question was considered and discussed first, by Nadler in [21]. Roughly speaking, Nadler considered two distinct results: a sequence contraction mappings which converges uniformly and a sequence contraction mappings that converges pointwise. This idea has been appreciated by a number of authors, see e.g.[10, 8, 24]. In this paper, we shall examine the existing results by

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considering the relationship in the partial metric spaces. We would like to emphasize that our results are not trivial. Because the limits of a convergent sequence in a partial metric space may be not unique.

2. Preliminaries

The notion of partial metric space was introduced by Matthews (see [19, 20]). The concept of partial metric space is obtained from metric space by replacing the condition $d(x,x)=0$ with the condition $d(x,x)\leqslant d(x,y)$ for all x,y in the definition of metric space. This notion has wide application potentials not only in the branches of mathematics, but also in the field of computer domain and semantics. Recently, many authors have focused on partial metric spaces and its topological properties, and generalized some fixed point theorems from the class of metric spaces to the class of partial metric spaces (see [2, 1, 3, 12, 9] and the references given therein).

Now, we shall recall some concepts.

Definition 2.1 (See e.g. [12, 19]). Let X be a nonempty set. The mapping $p: X\times X\to[0,\infty)$ is said to be a *partial metric* on X if for any $x,y,z\in X$ the following conditions hold true:

(P1) $x=y$ if and only if $p(x,x)=p(y,y)=p(x,y)$;

(P2) $p(x,x)\leqslant p(x,y)$;

(P3) $p(x,y)=p(y,x)$;

(P4) $p(x,z)\leqslant p(x,y)+p(y,z)-p(y,y)$.

The pair (X,p) is then called a *partial metric space* (in short PMS).

Let (X,p) be a partial metric space. Then, the functions $d_p, d_m: X\times X\to[0,\infty)$ given by

$$d_p(x,y)=2p(x,y)-p(x,x)-p(y,y)$$

and

$$d_m(x,y)=\max\{p(x,y)-p(x,x),p(x,y)-p(y,y)\}$$

are (usual) metrics on X. It is easy to check that d_p and d_m are equivalent. Note that each partial metric p on X generates a T_0-topology τ_p with a base of the family of open $p-$balls $\{B_p(x,\varepsilon): x\in X,\varepsilon>0\}$, where $B_p(x,\varepsilon)=\{y\in X: p(x,y)<p(x,x)+\varepsilon\}$.

Definition 2.2 (See e.g [12]). Let (X,p) be a partial metric space.

1) A sequence $\{x_n\}$ in X converges to $x\in X$ if and only if $p(x,x)=\lim_{n\to\infty}p(x_n,x)$;

2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if and only if $\lim_{n,m\to\infty}p(x_n,x_m)$ exists (and finite).

3) (X,p) is called to be *complete* if every Cauchy sequence $\{x_n\}$ in X converges to $x\in X$.

4) A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B(x_0, \delta)) \subset B(f(x_0), \varepsilon)$. A mapping f is continuous on X if it is continuous at every $x \in X$.

Theorem 2.3 (See e.g [12]). *Let (X, p) be a partial metric space and map $T : X \to X$. Then T is continuous on X if for every sequence $(x_n) \subset X$ and $x_n \to x$ then $\lim_{n\to\infty} p(Tx_n, Tx_n) = p(Tx, Tx)$.*

Example 2.4. Let $X = [0, +\infty)$ and define $p(x, y) = \max\{x, y\}$, for all $x, y \in X$. Then (X, p) is a complete partial metric space. It is clear that p is not a (usual) metric.

Proposition 2.5 (See e.g. [12]). *Let (X, p) be a partial metric space.*

1) A sequence $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if $\{x_n\}$ is a Cauchy sequence in (X, d_p).

2) (X, p) is complete if and only if (X, d_p) complete. Moreover,

$$\lim_{n\to\infty} d(x_n, x) = 0 \Leftrightarrow \lim_{n\to\infty} p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_m, x_n).$$

The following lemmas have an important technical role in the partial metric spaces.

Lemma 2.6 (See e.g. [12]). *Assume $x_n \to z$ as $n \to \infty$ in a PMS (X, p) such that $p(z, z) = 0$. Then $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ for every $y \in X$.*

Lemma 2.7 (See e.g. [12]). *Let (X, p) be a complete PMS. Then*

(A) *If $p(x, y) = 0$ then $x = y$,*

(B) *If $x \neq y$, then $p(x, y) > 0$.*

Lemma 2.8 (See e.g. [12]). *Assume $x_n \to x$ and $x_n \to y$ as $n \to \infty$ in a PMS (X, p) such that*

$$\lim_{n\to\infty} p(x_n, x_n) = p(x, x) = p(y, y).$$

Then $x = y$.

The following result is a generalization of the principle contraction in partial metric spaces.

Theorem 2.9 ([9]). *Let (X, p) be a complete partial metric space and $T : X \to X$ be self-mapping such that for all $x, y \in X$*

$$\begin{aligned} p(Tx, Ty) \leqslant \max\{&ap(x, y), bp(x, Tx), cp(y, Ty), \\ &d[p(x, Ty) + p(y, Tx)], p(x, x), p(y, y)\}, \end{aligned} \tag{1}$$

where $a, b, c \in [0, 1)$ and $d \in [0, \frac{1}{2})$. Then

1) $X_p = \left\{x \in X : p(x, x) = \inf\{p(y, y) : y \in X\}\right\}$ is nonempty.

2) There is a unique $u \in X_p$ such that $Tu = u$.

Remark 2.10. The Theorem 2.9 do not imply uniqueness of the fixed point of T. In the some following theorems and corollaries, under the somewhat stronger conditions, the uniqueness of the fixed point is hold.

Theorem 2.11 ([9]). *Let (X, p) be a complete partial metric space and $T : X \to X$ be self-mapping such that for all $x, y \in X$*

$$p(Tx, Ty) \leqslant M(x, y), \tag{2}$$

where

$$M(x, y) = \max\{ap(x, y), bp(x, Tx), cp(y, Ty), d[p(x, Ty) + p(y, Tx)], ep(x, x), fp(y, y)\},$$

and $a, b, c, 2d, e, f \in (0, 1)$. Then, there is a unique $z \in X$ such that $Tz = z$.

Corollary 2.12 ([9]). *Let (X, p) be a complete partial metric space and $T : X \to X$ be self-mapping such that for all $x, y \in X$*

$$\begin{aligned} p(Tx, Ty) \leqslant \max\{&ap(x, y), bp(x, Tx), cp(y, Ty), \\ &d[p(x, Ty) + p(y, Tx)], \frac{p(x, x) + p(y, y)}{2}\}, \end{aligned} \tag{3}$$

where $a, b, c \in [0, 1)$ and $d \in [0, \frac{1}{2})$. Then $X_p = \Big\{x \in X : p(x, x) = \inf\{p(y, y) : y \in X\}\Big\}$ is nonempty and T has a unique fixed point.

Corollary 2.13 ([9]). *Let (X, p) be a complete partial metric space and $T : X \to X$ be self-mapping such that for all $x, y \in X$*

$$p(Tx, Ty) \leqslant \max\{ap(x, y), bp(x, Tx), cp(y, Ty), d[p(x, Ty) + p(y, Tx)]\}, \tag{4}$$

where $a, b, c \in [0, 1)$ and $d \in [0, \frac{1}{2})$. Then $X_p = \Big\{x \in X : p(x, x) = \inf\{p(y, y) : y \in X\}\Big\}$ is nonempty and T has a unique fixed point.

The following result is the main corollary of Theorem 2.9. It is also the main result of Corollary [12].

Corollary 2.14 ([9, 12]). *Let (X, p) be a complete partial metric space and $T : X \to X$ be self-mapping such that for all $x, y \in X$*

$$p(Tx, Ty) \leqslant \max\{ap(x, y), p(x, x), p(y, y)\}, \tag{5}$$

where $a, b, c \in [0, 1)$ and $d \in [0, \frac{1}{2})$. Then

1) $X_p = \Big\{x \in X : p(x, x) = \inf\{p(y, y) : y \in X\}\Big\}$ is nonempty.

2) There is a unique $u \in X_p$ such that $Tu = u$.

The corollary is due to [3].

Corollary 2.15 ([9, 3]). *Let (X, p) be a complete partial metric space and $T : X \to X$ be self-mapping such that for all $x, y \in X$*

$$p(Tx, Ty) \leqslant ap(x, y) + bp(Tx, x) + cp(y, Ty) + d(p(Tx, y) + p(x, Ty)) \tag{6}$$

where $a + b + c + 2d < 1$, and $a, b, c, d \geqslant 0$. Then T has a unique fixed point.

We also get another result from Theorem 2.11. It is a generalization of Corollary 2.15.

Corollary 2.16. ([9]) *Let (X, p) be a complete partial metric space and $T : X \to X$ be self-mapping such that for all $x, y \in X$*

$$\begin{aligned} p(Tx, Ty) \leqslant & ap(x, y) + bp(Tx, x) + cp(y, Ty) \\ & + d[p(Tx, y) + p(x, Ty)] + ep(x, x) + fp(y, y) \end{aligned} \tag{7}$$

where $a + b + c + 2d + e + f < 1$, and $a, b, c, d, e, f \geqslant 0$. Then T has a unique fixed point.

Definition 2.17. Let (X, p) be a partial metric space, maps $T_n : X \to X, n \in \mathbb{N}$ and the $(a_n) \subset X$. The sequence (a_n) is to be called a *fixed points sequence* of (T_n) if $T_n(a_n) = a_n$ for every $n \in \mathbb{N}$.

Next, we shall introduce some concepts on the convergence of sequences of mappings in partial metric spaces.

Definition 2.18. Let (X, p) be a partial metric space and maps $T_n : X \to X, n \in \mathbb{N}$, and a map $T : X \to X$.

1) The sequence (T_n) converges pointwise to T if for each $x \in X$

$$\lim_{n\to\infty} p(T_n x, Tx) = p(Tx, Tx).$$

2) The sequence (T_n) converges pointwise to T in the sense metric d_p if for each $x \in X$

$$\lim_{n\to\infty} d_p(T_n x, Tx) = 0.$$

3) The sequence (T_n) converges uniformly to T if

$$\lim_{n\to\infty} \sup_{x\in X} |p(T_n x, Tx) - p(Tx, Tx)| = 0.$$

We get some simple facts.

Lemma 2.19. *Let (X, p) be a partial metric space and sequences $(x_n), (y_n) \subset X$, and $x, y \in X$. If $x_n \to x$ and $y_n \to y$ in the metric space (X, d_p) then $p(x_n, y_n) \to p(x, y)$.*

Proof. Since $x_n \to x$ and $y_n \to y$ in the metric space (X, d_p), we have

$$d_p(x_n, x) = \max\{p(x_n, x) - p(x, x), p(x_n, x) - p(x_n, x_n)\} \to 0$$

and

$$d_p(y_n, y) = \max\{p(y_n, y) - p(y, y), p(y_n, y) - p(y_n, y_n)\} \to 0$$

as $n \to \infty$. We have

$$\begin{aligned} p(x_n, y_n) &\leqslant p(x_n, x) + p(x, y_n) - p(x, x) \\ &\leqslant p(x_n, x) + p(x, y) + p(y, y_n) - p(y, y) - p(x, x). \end{aligned}$$

It follows that

$$p(x_n, y_n) - p(x, y) \leqslant [p(x_n, x) - p(x, x)] + [p(y, y_n) - p(y, y)] \to 0$$

as $n \to \infty$. On the other hand

$$\begin{aligned} p(x, y) &\leqslant p(x_n, x) + p(x_n, y) - p(x_n, x_n) \\ &\leqslant p(x_n, x) + p(x_n, y_n) + p(y, y_n) - p(y_n, y_n) - p(x_n, x_n). \end{aligned}$$

We obtain

$$p(x, y) - p(x_n, y_n) \leqslant [p(x_n, x) - p(x_n, x_n)] + [p(y, y_n) - p(y_n, y_n)] \to 0$$

as $n \to \infty$.

Combining the above inequalities, we arrive at $|p(x_n, y_n) - p(x, y)| \to 0$ as $n \to \infty$. The lemma is proved. □

Lemma 2.20. *Let (X, p) be a partial metric space and T be a self-mapping on X. If (T_n) is a sequence of continuous self-mappings on X and (T_n) converges uniformly to T then T is continuous.*

Proof. Suppose $(x_k) \subset X$ be a sequence that converges to a. For each $\varepsilon > 0$ and $n = 1, 2, ...$, there exists $k_0 = k_0(n)$ such that

$$|p(T_n x_k, T_n x_k) - p(T_n a, T_n a)| < \varepsilon$$

for all $k \geqslant k_0$. Since (T_n) converges uniformly to $T : X \to X$, there is a $n_0 = n_0(\varepsilon)$ such that

$$|p(T_n x, Tx) - p(Tx, Tx)| < \varepsilon$$

for every $x \in X$ and $n \geqslant n_0$. Hence, if $n \geqslant n_0$ then

$$|p(T_n x_k, Tx_k) - p(Tx_k, Tx_k)| < \varepsilon$$

and

$$|p(T_n a, Ta) - p(Ta, Ta)| < \varepsilon.$$

This implies that, $k \geqslant k_0(n_0)$

$$\begin{aligned} &|p(Tx_k, Tx_k) - p(Ta, Ta)| \\ &\leqslant |p(T_n x_k, Tx_k) - p(Tx_k, T_k)| + |p(T_n x_k, T_n x_k) - p(T_n a, T_n a)| \\ &\quad + |p(T_n a, Ta) - p(Ta, Ta)| < 3\varepsilon \end{aligned}$$

holds for every $n \geqslant n_0$. Thus $\lim_{k\to\infty} p(Tx_k, Tx_k) = p(Ta, Ta)$ and so that T is continuous. □

Definition 2.21. Let (X, p) be a partial metric space and (p_n) be a partial metric sequence on X. The sequence (p_n) is called to be uniformly convergent to p if

$$\sup_{x,y\in X} |p_n(x, y) - p(x, y)| \to 0$$

as $n \to \infty$.

3. The main results

We begin the section at the convergence of fixed points of a sequence of continuous mappings in partial metric spaces.

Theorem 3.1. *Let (X, p) be a partial metric space. Suppose that (T_n) is a sequence of continuous mapping from X to X and (a_n) is the fixed points sequence of (T_n). Then, if (T_n) uniformly converges to mapping T on X and there exists $a \in X$ such that*

$$p(a, a) = \lim_{n\to\infty} p(a_n, a) = \lim_{n\to\infty} p(a_n, a_n) = \lim_{n\to\infty} p(a_n, Ta)$$

then a is a fixed point of T.

Proof. Since (a_n) is a fixed points sequence of (T_n) and $\lim_{n\to\infty} a_n = a$, we have that

$$\lim_{n\to\infty} p(T_n a_n, T_n a_n) = \lim_{n\to\infty} p(a_n, a_n) = p(a, a).$$

In view of Lemma 2.20 and (T_n) is a sequence of continuous mapping, we infer from T_n converges uniformly to T that T is continuous on X. For $\varepsilon > 0$ arbitrary, we can seek n_1 such that

$$|p(Ta_n, Ta_n) - p(Ta, Ta)| < \frac{\varepsilon}{2} \tag{8}$$

for every $n \geq n_1$. On the other hand, since (T_n) converges uniformly to T, we have that

$$\lim_{n\to\infty} \sup_{x\in X} |p(T_n x, Tx) - p(Tx, Tx)| = 0$$

for every $x \in X$. Hence, there exists n_2 such that

$$|p(T_n x, Tx) - p(Tx, Tx)| < \frac{\varepsilon}{2} \tag{9}$$

for every $n \geqslant n_2$ and $x \in X$. Combining (8) and (9), we can deduce that

$$\begin{aligned} |p(Ta, a_n) - p(Ta, Ta)| &= |p(Ta, T_n a_n) - p(Ta, Ta)| \\ &\leqslant |p(T_n a_n, Ta_n) - p(Ta_n, Ta_n)| + |p(Ta_n, Ta_n) - p(Ta, Ta)| < \varepsilon \end{aligned}$$

for every $n \geqslant n_0 = \max\{n_1, n_2\}$. This implies that $\lim_{n\to\infty} p(a_n, Ta) = p(Ta, Ta)$, or $\lim_{n\to\infty} a_n = Ta$. Combining with the hypothesis, we have

$$p(a, a) = \lim_{n\to\infty} p(a_n, a_n) = p(Ta, Ta).$$

By Lemma 2.8, we have $a = Ta$. The proof is finished. □

Corollary 3.2. ([21]) *Let (X, d) be a metric space. Suppose that (T_n) is a sequence of continuous mappings from X to X and (a_n) is a fixed point sequence of (T_n). If (T_n) converges uniformly T on (X, d) and $\lim_{n\to\infty} a_n = a$ then a is a fixed point of T.*

The following example shows that if we replace uniformly convergence by pointwise convergence then the conclusion of Theorem 3.2 is no longer correct.

Example 3.3. Let $X = [0, +\infty)$ and the partial metric be defined

$$p(x, y) = \max\{x, y\},$$

for all $x, y \in X$. We consider the sequence of maps $T_n : X \to X$ that is defined in the following way:

$$T_n x = \begin{cases} x + 1 & \text{if } \ x \notin (0, \frac{1}{n}) \\ (1 - 2n)x + 1 & \text{if } \ 0 < x \leqslant \frac{1}{2n} \\ (2n - 1)x - 1 + \frac{1}{n} & \text{if } \ \frac{1}{2n} < x < \frac{1}{n}. \end{cases}$$

for each $n \in \mathbb{N}$. It is easily seen that T_n is continuous on $\mathbb{X}$ and $\frac{1}{2n}$ is a fixed point of T_n for each $n \in \mathbb{N}$. Moreover, it is easy to check that (T_n) converges pointwise to the map $T(x) = x + 1$ for all $x \in \mathbb{X}$. However $0 = \lim_{n\to\infty} \frac{1}{2n}$ is not a fixed point of T. In fact, T has no fixed point and (T_n) does not converge uniformly to T. Indeed, we have

$$\sup_{x\in\mathbb{R}} |p(T_n x, Tx) - p(Tx, Tx)| \geqslant \sup_{x\in(0,\frac{1}{2n})} |(x + 1) - ((1 - 2n)x + 1)| = 1$$

for all n. This proves that (T_n) is not uniformly convergent to T.

Next, we study the convergence of the sequence of fixed points of generalized contractive mappings in partial metric spaces.

Theorem 3.4. *Let (X, p) be a complete partial metric space. Let (T_n) be a sequence of self-mappings on X such that*

$$p(T_n x, T_n y) \leqslant \max\Big\{a_n p(x, y), b_n p(x, T_n x), c_n p(y, T_n y), d_n[p(x, T_n y) + p(y, T_n x)], p(x, x), p(y, y)\Big\}, \tag{10}$$

holds for every $x, y \in X$ and for each $n = 1, 2, \ldots$, where $a_n, b_n, c_n \in [0, 1)$; $d_n \in (0, \frac{1}{2})$ for all $n = 1, 2, \ldots$. Suppose that (T_n) converges pointwise to T in the sense metric d_p and

$\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, $\lim_{n\to\infty} c_n = c$, $\lim_{n\to\infty} d_n = d$ with $a, b, c \in [0, 1)$ and $d \in [0, \frac{1}{2})$. Then there are

$$z_n \in X_p = \Big\{x \in X : p(x, x) = \inf\{p(y, y) : y \in X\}\Big\}$$

such that $T_n(z_n) = z_n$ for each $n = 1, 2, ...,$ $\lim_{n\to\infty} z_n = z$ and z is a fixed point of T.

Proof. Firstly, it follows from Theorem 2.9 and (15) that $X_p \neq \emptyset$. Moreover, there exists a unique $z_n \in X_p$ such that $T_n(z_n) = z_n$ for each $n = 1, 2, ...$ We shall show that $\lim_{n\to\infty} z_n = z$ and z is a fixed point of T. For each $x, y \in X$, under the convergence of (T_n) to T on X, by applying Lemma 2.19 and letting $n \to \infty$, we obtain that

$$\begin{aligned} p(Tx, Ty) \leqslant \max\Big\{&ap(x, y), bp(x, Tx), cp(y, Ty), \\ &d[p(x, Ty) + p(y, Tx)], p(x, x), p(y, y)\Big\}, \end{aligned} \tag{11}$$

for every $x, y \in X$. Applying Theorem 2.9 again, we get the unique $z \in X_p$ such that $T(z) = z$. It remains to check that $\lim_{n\to\infty} z_n = z$. To do this, we shall show that

$$\lim_{n\to\infty} p(z_n, z) = p(z, z).$$

Set $\rho_p = \inf\{p(y, y) : y \in X\}$. It follows from $(z_n) \subset X_p$ and $z \in X_p$ that

$$p(z_n, z_n) = p(z, z) = \rho_p, \tag{12}$$

for all $n = 1, 2, ...$ In view the axiom P4), we have

$$p(z_n, z) = p(z_n, T_n z) + p(T_n z, z) - p(T_n z, T_n z) \tag{13}$$

for each $n = 1, 2, ...$ By the condition (10), we obtain

$$\begin{aligned} p(z_n, T_n z) &= p(Tz_n, T_n z) \\ &\leqslant \max\Big\{a_n p(z_n, z), b_n p(z_n, T_n z), c_n p(T_n z_n, z_n), \\ &\quad d[p(z, T_n z_n) + p(z_n, T_n z)], p(z_n, z_n), p(z, z)\Big\}, \end{aligned} \tag{14}$$

Letting $n \to \infty$, we arrive at

$$\lim_{n\to\infty} p(z_n, T_n z) \leqslant \rho_p$$

Combining with (13), we have

$$\lim p(z_n, z) \leqslant \rho_p.$$

From $\lim_{n\to\infty} p(z_n, z) \geqslant \rho_p$, we can deduce that

$$\lim_{n\to\infty} p(z_n, z) = \rho_p.$$

Hence

$$\lim_{n\to\infty} p(z_n, z) = \lim_{n\to\infty} p(z_n, z_n = p(z, z) = \lim \rho_p.$$

This proves that z_n converges to z. □

The following corollary is derived from $a_n = a, b_n = b, c_n = c$ and $d_n = d$ for all $n = 1, 2, ...$

Corollary 3.5. *Let (X, p) be a complete partial metric space. Let (T_n) be a sequence of self-mappings on X such that*

$$p(T_n x, T_n y) \leqslant \max\Big\{ap(x,y), bp(x,T_nx), cp(y,T_ny), d[p(x,T_ny)+p(y,T_nx)], p(x,x), p(y,y)\Big\}, \quad (15)$$

holds for every $x, y \in X$ and for each $n = 1, 2, ...,$ where $a, b, c \in [0, 1)$ and $d \in (0, \frac{1}{2})$. Suppose that (T_n) converges pointwise to T in the sense metric d_p. Then there are

$$z_n \in X_p = \Big\{x \in X : p(x,x) = \inf\{p(y,y) : y \in X\}\Big\}$$

such that $T_n z_n = z_n$ for each $n = 1, 2, ..., \lim_{n\to\infty} z_n = z$ and z is a fixed point of T.

In view Corollary 2.13 and $q = \max\{a, b, c, 2d\}$, we get the following corollary.

Corollary 3.6. *Let (X, p) be a complete partial metric space. Let (T_n) be sequence of self-mappings on X such that*

$$p(T_n x, T_n y) \leqslant q \max\Big\{p(x,y), p(x,T_nx), p(y,T_ny), \frac{p(x,T_ny)+p(y,T_nx)}{2}\Big\} \quad (16)$$

holds for all $x, y \in X$ and for each $n = 1, 2, ...,$ where $0 < q < 1$. If (T_n) converges pointwise to mapping $T : X \to X$ in the sense metric d_p then there exists $(a_n) \subset X$ such that $T_n(a_n) = a_n$ for each $n = 1, 2, ..., \lim_{n\to\infty} a_n = a$ and a is a fixed point of T.

Corollary 3.7. *Let (X, p) be a complete partial metric space. Let (T_n) be sequence of self-mappings on X such that*

$$p(T_n x, T_n y) \leqslant ap(x,y) + bp(x,T_nx) + cp(y,T_ny) + dp(x,T_ny) + ep(y,T_nx)\} \quad (17)$$

for all $x, y \in X$, where $0 < a, b, c, 2d, 2e < 1$. If (T_n) converges pointwise to mapping $T : X \to X$ in the sense metric d_p then $\lim_{n\to\infty} a_n = a$ exists and is a fixed point of T.

Proof. Put $q = \max\{a, b, c, 2d, 2e\}$. Then $0 < q < 1$ and

$$\begin{aligned} p(T_n x, T_n y) &\leqslant ap(x,y) + bp(x,T_nx) + cp(y,T_ny) + dp(x,T_ny) + ep(y,T_nx)\} \\ & q\max\{p(x,y), p(x,T_nx), p(y,T_ny), \frac{p(x,T_ny)+p(y,T_nx)}{2}\}. \end{aligned} \quad (18)$$

□

Corollary 3.8. *Let (X, p) be a complete partial metric space. Let (T_n) be a sequence of self-mappings on X such that*

$$p(T_n x, T_n y) \leqslant qp(x,y),$$

for all $x, y \in X$, where $q \in (0,1)$. If (T_n) converges pointwise to T in the sense metric d_p then T_n has a unique fixed point a_n for each $n = 1, 2, ..$ and $\lim_{n\to\infty} a_n = a$ is a unique fixed point of T.

Remark 3.9. In the above results, we would like to emphasize that the pointwise convergence in the sense metric d_p can not be replaced by pointwise convergence in the partial metric p. The following example gives an illustration.

Example 3.10. Let $X := [0,1] \cup [2,3]$ and define $p : X \times X \to \mathbb{R}$

$$p(x,y) = \begin{cases} |x-y| & \text{if} \quad \{x,y\} \subset [0,1] \\ \max\{x,y\} & \text{if} \quad \{x,y\} \cap [2,3] \neq \emptyset. \end{cases}$$

It is easy to check that (X,p) is a complete partial space. Let $T_n : X \to X$ be defined by

$$T_n x = \begin{cases} \dfrac{1}{n+1} & \text{if} \quad 0 \leqslant x < 1 \\ \dfrac{n}{n+1} & \text{if} \quad x = 1 \quad \text{or} \quad 2 \leqslant x \leqslant 3. \end{cases}$$

For each $n = 1, 2, ...$, it is easy to see that $\dfrac{1}{n+1}$ is the unique fixed point of T_n and (T_n) converges pointwise to T defined by

$$Tx = \begin{cases} 0 & \text{if} \quad 0 \leqslant x < 1 \\ 2 & \text{if} \quad x = 1 \quad \text{or} \quad 2 \leqslant x \leqslant 3. \end{cases}$$

and $x = 1$ is a fixed point of T. But $x_n = \dfrac{1}{n+1}$ does not converge to $x = 1$. In fact, (T_n) converges pointwise to T but T_n does not converge pointwise to T in the sense metric d_p. Indeed, if $x \in [2,3]$ or $x = 1$ then

$$p(T_n x, Tx) = \max\{T_n x, Tx\} = \max\{\frac{n}{n+1}, 2\} = 2 \to 2 = p(Tx, Tx)$$

as $n \to \infty$. If $0 \leqslant x < 1$ then

$$p(T_n x, Tx) = |\frac{1}{n+1} - 0| \to 0 = |0 - 0| = p(Tx, Tx)$$

as $n \to \infty$. Hence, T_n converges pointwise to T. If $x = 1$ then

$$d_p(T_n 1, T1) = |\frac{n}{n+1} - 2| \to 1 \neq 0$$

as $n \to \infty$. This shows that (T_n) does not converges pointwise to T.

The following Theorem is based on Ivanov's ideas ([17]).

Theorem 3.11. *Let (X,p) be a complete partial metric space, $0 < q < 1$ and (p_n) is a sequence of partial metrics on X. Suppose that $T_n : X \to X, n \in \mathbb{N}$ is a sequence of mappings with a fixed points sequence (a_n) satisfying:*

1) T_n is q-contraction in partial metric space (X, p_n);

2) (p_n) converges uniformly to partial metric p;

3) (T_n) converges pointwise to T in the sense metric d_p.

Then, there exists $\lim\limits_{n\to\infty} a_n = a$ and a is a fixed point of T.

Proof. We first show that T is $q-$contraction in the partial metric space (X,p). In fact, for each $x, y \in X$ and $\varepsilon > 0$ arbitrarily. Since (T_n) converges pointwise to T, there exists n_0 such that

$$p(Tx, Ty) \leqslant p(T_n x, T_n y) + \frac{\varepsilon}{3}$$

for all $n \geqslant n_0$. On the other hand, since (p_n) converges uniformly to p, we can find n_1 (independent of x, y) such that

$$|p_n(x,y) - p(x,y)| \leqslant \frac{\varepsilon}{3}$$

for all $n \geqslant n_1$. Then, by $n \geqslant \max\{n_0, n_1\}$, we get

$$\begin{aligned} p(Tx, Ty) &\leqslant p(T_n x, T_n y) + \frac{\varepsilon}{3} \\ &\leqslant p_n(T_n x, T_n y) + \frac{2\varepsilon}{3} \\ &\leqslant qp_n(Tx, Ty) + \frac{\varepsilon}{3} \leqslant qp(x,y) + \varepsilon \end{aligned}$$

for every $x, y \in X$. As $\varepsilon > 0$ arbitrary, we infer that $p(Tx, Ty) \leqslant qp(x,y)$. Thus, T is a $q-$contraction.

Now, since (X,p) is complete, there is a unique fixed point $a \in X$ of T. From

$$p(a,a) = p(Ta, Ta) \leqslant qp(a,a)$$

this leads $p(a,a) = 0$. We have

$$p_n(a, a_n) = p_n(a, T_n a_n) \leqslant p_n(a, T_n a) + p_n(T_n a, T_n a_n) \leqslant p_n(a, T_n a) + qp_n(a, a_n)$$

for all n and therefore

$$p_n(a, a_n) \leqslant \frac{p_n(a, T_n a)}{1-q}.$$

For all $n \geqslant n_1$, we obtain that

$$p(a, a_n) - \frac{\varepsilon}{3} \leqslant \frac{p(a, T_n a) + \varepsilon/3}{1-q}.$$

This yields that

$$p(a, a_n) \leqslant \frac{p(a, T_n a) + \varepsilon/3}{1-q} + \frac{\varepsilon}{3}.$$

Since $\lim_{n\to\infty} p(a, T_n a) = p(a, Ta) = p(a, a) = 0$, we can find n_2 such that $p(a, T_n a) \leqslant \varepsilon$ for every $n \geqslant n_2$. Then, by $n \geqslant \max\{n_1, n_2\}$, we get

$$p(a, a_n) \leqslant \frac{4\varepsilon}{3(1-q)} + \frac{\varepsilon}{3}.$$

As ε arbitrary, we have $\lim_{n\to\infty} p(a, a_n) = 0 = p(a, a)$. This yields that a_n converges to a. Moreover, it follows from $p(a, Ta) = 0$ that $Ta = a$. □

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