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ON THE CHARACTERIZATION OF HOMOGENEOUS POLYNOMIAL CONVEXITY SETS IN $\mathbb{C}^n$

Một số đặc trưng của tập lồi đa thức thuần nhất trong $\mathbb{C}^n$

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ABSTRACT

In this note, we are concerned with the homogeneous polynomial convexity of compact sets in $\mathbb{C}^n$. Precisely, we give an characterization in the Oka-Stolzenberg's sense of homogeneous polynomial convexity sets in $\mathbb{C}^n$

Keywords: *Polyniomial convexity set*

TÓM TẮT

Trong bài viết này, chúng tôi khảo sát tính lồi đa thức thuần nhất của tập compact trong $\mathbb{C}^n$. Cụ thể hơn, chúng tôi đưa ra một số đặc trưng theo nghĩa Oka-Stolzenberg đối với tập lồi đa thức thuần nhất trong $\mathbb{C}^n$

Từ khóa: *tập lồi đa thức*

1. Introduction and Statement Result

Denote by $\mathscr{H}$ and $\mathscr{S}$, respectively, the set of homogeneous polynomials and set of homogeneous polynomials in $\mathbb{C}^n$ of the forms

$$p(z)=\prod_{k=1}^{m}\left(\alpha_1^k z_1+...+\alpha_n^k z_n\right),\ z=(z_1,...,z_n)\in\mathbb{C}^n,\ \text{for all } m=1,2,...$$

Let K be a compact subset of $\mathbb{C}^n$. $\hat{K}$ denotes the polynomially convex hull of K, i.e.,

$$\hat{K}=\left\{z\in\mathbb{C}^n :|\ p(z)|\le \max_K|\ p|\ \text{for every polynomial } p \text{ in } \mathbb{C}^n\right\}.$$

We say that K is polynomially convex if $\hat{K}=K$. We call the following set the homogeneous polynomially convex hull of K:

$$\hat{K}_{hom}=\left\{z\in\mathbb{C}^n :|\ p(z)|\le \max_K|\ p|\ \text{for every } p\in\mathscr{H}\right\}.$$

K is called a homogeneous polynomially convex set if $\hat{K}=K_{hom}$. A similar definition of class $\mathscr{S}$ was introduced in [2]:

$$\hat{K}_S=\left\{z\in\mathbf{C}^n :|\ p(z)|\le \max_K|\ p|\ \text{for every } p\in\mathscr{S}\right\}.^{1}$$

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The interest in polynomial convexity stems from the celebrated Oka-Weil approximation theorem, which states that holomorphic functions near a compact polynomially convex subset of $\mathbb{C}^n$ can be uniformly approximated by polynomials in $\mathbb{C}^n$. Moreover, it is well known that K and $\hat{K}$ can respectively be identified with the spaces of maximal ideals of $C(K)$ and $P(K)$, where $C(K)$ is the algebra of continuous and complex valued functions on K and $P(K)$ is uniform algebra of polynomials on K. A compact $K \subset \mathbf{C}$ is polynomially convex if it is $\mathbb{C} \setminus K$ connected. In higher dimensions, no such topological characterization of polynomially convex sets is available, and it is usually difficult to determine whether a given compact subset is polynomially convex. There are multiple approaches to characterizing the polynomial hull, (see [11]). One classical characterization is the Oka-Stolzenberg characterization, which describes the polynomial hull in the curve of algebraic hypersurfaces (see [12]). Note that $K \subset H \subset \hat{K} \subset K_{hom} \subset K_S$. Moreover, these inclusions may be proper some properties of hulls by means of topological reasonings.

Homogeneous polynomial convexity also approaches projective capacity (see [2]) and the projective hull (see [5,10]). Even in case of polynomial convexity, it is difficult to check whether a given compact subset is homogeneous polynomially convex. Exploration of the homogeneous polynomially convex hull of compact sets in this feild has been limited so far. In [8,9], Khudaiberganov explored the homogeneous polynomially convex hull of a union of balls in $\mathbf{C}^n$.

If K is a circled compact set of $\mathbb{C}^n$ (i.e., S^1 invariant), then $\hat{K} = \hat{K}_{hom}$ ([2,5]). Precisely, if K is invariant under the map $T_\varphi(z_1, z_2, ..., z_n) = \left(e^{i\varphi} z_1, .., e^{i\varphi} z_n\right)$ for all $0 \le \varphi < 2\pi$, then $\hat{K} = \hat{K}_{hom}$. The following example shows that $\hat{K} = \hat{K}_{hom}$ does not imply that K is a circled set.

Example 1.1. Let

$$K = \{z \in \mathbb{C} : |z| \le 2\} \setminus \{z \in \mathbb{C} : |z| < 1, \ \mathrm{Im}\, z > 0\}.$$

It is easy to see that K is not S^1 invariant, and $\hat{K} = \hat{K}_{\mathrm{hom}} = \{z \in \mathbb{C} : |z| \le 2\}$

2. A Characterization of Homogeneous Polynomial Convexity

Let $F : [s, \infty) \times \mathbb{C}^n \to \mathbb{C}$ be a continuous function such that $F_t(z) := F(t, z)$ is a nonconstant polynomial for each $t \in [0, \infty)$. We set $H_t = \{z \in \mathbb{C}^n : F(t, z) = 0\}$,

for each $t \in [s, \infty)$. Then, $t \mapsto H_t$, (H_t) is a curve of algebraic hypersurfaces $\mathbb{C}^n$, and is denoted by (H_t). If the distance $d(H_t, 0) \to \infty$ as $t \to \infty$, then (H_t) is said to connect the initial hypersurface to the hyperplane at infinity. If there exists $t_0 > s$ such that $H_{t_0} \cap X \ne \varnothing$, then (H_t) passes through X.

The following is the Oka-Stolzenberg characterization of a polynomially convex hull:

Theorem 2.1. ([12]) *If X is a compact set of $\mathbb{C}^n$, then $\hat{X}$ equal to the set of points $a \in \mathbb{C}^n$ such that every curve of the*

algebraic hypersurfaces that passes through a and joins the initial hypersurfaces to the hyperplane at infinity, also passes through X.

Every curve in $\mathbb{C}$ joins a point to infinity that is the curve of an algebraic hypersurface. Indeed, if L is a curve in $\mathbb{C}$ with the parametric equation $z = z(t)$, $a \le t \le +\infty$, then $F : [a, +\infty) \times \mathbb{C}$ is defined by $F(t,z) = z - z(t)$ that generates the curve $(H_t) = L$. The algebraic hypersurfaces are points on the curve. This fact leads to the following corollary:

Corollary 2.2. *Let X be a compact subset of $\mathbb{C}$. Then, the polynomially convex hull $\hat{X}$ is equal to the union of X and the bounded connected component of $\mathbb{C} \setminus X$. In particular, if $\mathbb{C} \setminus X$ is connected, then X is polynomially convex.*

In higher dimensions, no topological conditions are available such that a compact subset K is polynomially convex.

Definition 2.3. Let (H_t) be a curve of algebraic hypersurfaces, where

$$H_t = \{z \in \mathbb{C}^n : F(t,z) = 0\}.$$

(H_t) is said to be a curve of homogeneous algebraic hypersurfaces if there is a constant b such that $F(s,z) + b$ is a homogeneous polynomial.

We now state the Oka-Stolzenberg characterization of a homogeneous polynomial hull.

Theorem 2.4. *Let K be a compact set of $\mathbb{C}^n$. A point $a \in \mathbb{C}^n$ belongs to $\hat{K}_{hom}$ if and only if every curve of homogeneous algebraic hypersurfaces that passes through a and joins the initial hypersurface to the hyperplane at infinity, also passes through S^1K.*

Proof. Let $a \notin \hat{K}_{hom}$. We can then find a homogeneous p such that $p(a) = 1$ and $\| p \|_K < 1$. Consider $F : [1, \infty) \times \mathbb{C}^n \to 0$, defined by

$$F(z,t) = p(z) - t,$$

for every $z \in \mathbb{C}^n$ and $t \in [1, \infty)$. We get a curve of the homogeneous algebraic hypersurfaces (H_t) defined by F such that it passes through a, joins the initial hypersurface to the hyperplane at infinity, and does not pass through S^1K. Indeed, it follows from $F(a,1) = p(a) = 1$ that $a \in \hat{K} \cap H_1$.

We show that (H_t) joins the initial hypersurface to the hyperplane at infinity. Suppose that $p(z) = \sum_{k_1 + k_2 + \ldots + k_n = k} a_k z_1^{k_1} z_2^{k_2} \ldots z_n^{k_n}$,

where $k = (k_1, .., k_n)$ is a multi-index. Then, there exists $M > 0$ such that $|p(z)| \le M |z|^{|k|}$

for all $z = (z_1, ..., z_n) \in \mathbb{C}^n$, where $|z| = \sqrt{|z_1|^2 + \ldots + |z_n|^2}$ and $k_1 + k_2 + \ldots + k_n = |k|$.

Therefore, $|p(u)| = t \le M |u|^k$, for every $u \in H_t$. Hence, if $t \to \infty$, then $|u| \to \infty$ for all $u \in H_t$. This yields

$$d(H_t, 0) = \inf \{|u| : u \in H_t\} \to \infty.$$

Suppose that (H_t) passes through S^1K. Then, there exists $t_0 > 1$ and $z \in H_{t_0} \cap S^1K$. It follows that

$$F(t_0, z) = p(z) - t_0 = 0 \Leftrightarrow t_0 = p(z).$$

Since $z \in S^1K$, $z = \lambda a$, $a \in K$, $\lambda \in S^1$, and $|p(z)| = |p(\lambda a)| = |\lambda^n| \, |p(a)| = |p(a)| \leq \|p\|_K < 1$.

This yields a contradiction.

Let $a \in \hat{K}_{hom}$. Suppose that there is a curve of homogeneous algebraic hypersurfaces that passes through a, joins the initial hypersurface to the hyperplane at infinity, and does not pass through S^1K. By the Oka-Stolzenberg characterization of the polynomial hull, we can deduce that $a \notin S^1K$. We get a contradiction with $\hat{K}_{hom} \subset S^1K$. The theorem is thus proved. □

We get the following corollary:

Corollary 2.5. *Let K be a compact subset of $\mathbb{C}$. Then, K is homogeneous polynomially convex if and only if $\mathbb{C} \setminus S^1K$ is connected.*

The following example shows that $K = \hat{K} \neq \hat{K}_{hom}$. Hence, $\mathbb{C} \setminus K$ is connected, which does not imply the homogeneous polynomial convexity of K.

Example 2.6. Let

$$K = \left\{ z \in \mathbb{C} : |z| \leq 1, -\frac{\pi}{4} \leq \arg z \leq \frac{\pi}{4} \right\} \cup \left\{ z \in \mathbb{C} : |z| \leq 1, \frac{3\pi}{4} \leq \arg z \leq \frac{5\pi}{4} \right\}.$$

It follows from $\mathbb{C} \setminus K$ being connected that $\hat{K} = K$.

Let $z_0 \in \mathbb{C}$ and $|z_0| \leq 1$. Suppose that p is a homogeneous polynomial in $\mathbb{C}$. Then,

$$p(z) = az^m$$

for all $z \in \mathbb{C}$, where $m \in \mathbb{N}$. We have

$$|p(z_0)| = |a| \, |z_0|^m \leq |a| = \max_{z \in K} |p(z)|.$$

This implies that $z_0 \in \hat{K}_{hom}$.

If $|z_0| > 1$, we consider the homogeneous polynomial $p(z) = e^{-\arg z_0} z$. We have

$$p(z_0) = |z_0| > 1 = \max_{z \in K} |p(z)|.$$

It follows that $z_0 \notin \hat{K}_{hom}$. Therefore,

$$\hat{K}_{\text{hom}} = \{ z \in \mathbb{C} : |z| \leq 1 \} \neq K$$

Example 1.1 shows that $\hat{K} = \hat{K}_{hom}$ does not imply that K is S^1 invariant. The following proposition states that this holds for $\hat{K}$:

Proposition 2.7. *Let* K *be a compact subset of* $\mathbb{C}$. *Then,* $\hat{K} = \hat{K}_{hom}$ *if and only if* $\hat{K}$ *is* S^1 *invariant.*

Proof. We first show that $S^1\hat{K} \subset \hat{K}_{hom}$. Let $z_0 = \lambda b$, $\lambda \in S^1$ and $b \in \hat{K}$. Suppose that p is a homogeneous polynomial in $\mathbb{C}$. Then,

$$p(z) = az^m$$

for all $z \in \mathbb{C}$, where $m \in \mathbb{N}$. Therefore

$$|p(z_0)| = |a(\lambda b)^m| = |a||\lambda||b|^m = |a||b|^m = |p(b)| \leq \max_{z \in \hat{K}} |p(z)|.$$

Hence, $z_0 \in \hat{K}_{hom}$, so that $S^1\hat{K} \subset \hat{K}_{hom}$. It then follows from $\hat{K} = \hat{K}_{hom}$ that $\hat{K}$ is S^1 invariant.

Suppose that $\hat{K}$ is S^1 invariant. We have $S^1\hat{K} = \hat{K}$. Since $(S^1K) \subset S^1\hat{K}$, we obtain $\hat{K} = \hat{K}_{hom}$. □

Remark 2.8. We emphasize that the previous statement does not hold in higher dimensions

3. Conclusions

Homogeneous polynomial convexity of compact sets in $\mathbb{C}^n$ is always a great challenge, especially characterization in the Oka-Stolzenberg's sense of homogeneous polynomial convexity sets in $\mathbb{C}^n$. In previous studies, mathematicians have shown some characteristics of polynomial convex hulls in a number of forms, of which, classical characterization is the Oka-Stolzenberg characterization, which describes the polynomial hull in the curve of algebraic hypersurfaces. This paper presents Oka-Stolzenberg characterization of a homogeneous polynomial hull, the polynomially convex hull $\hat{X}$ is equal to the union of X and the bounded connected component of $\mathbb{C} \setminus X$. In particular, if $\mathbb{C} \setminus X$ is connected, then X is polynomially convex. We emphasize that the previous statement does not hold in higher dimensions.

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