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INVERSE SOURCE PROBLEM FOR THE ADVECTION EQUATION WITH DISCRETE RANDOM NOISE

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ABSTRACT

In this article, we deal with an inverse problem of determining an unknown time-dependent source in a one-dimensional advection equation, where the aid of an extra measurement of contaminant density at an internal point. Specially, the problem is considered with the statistical discrete data. In general, this problem is ill-posed, therefore the trigonometric method in nonparametric regression associated with the truncated expansion method is proposed to construct the regularized source. Moreover, the convergence rate of the regularized source is also obtained. Eventually, a numerical experiment is given to illustrate the effectiveness and flexibility of our proposed method.

Keywords: *inverse source problem, nonparametric regression, trigono-metric method, statistical discrete data.*

AMS Subject Classification: 35J05, 35R25, 35K05, 47A52, 62G08.

1. Introduction

Until now, inverse source problems are important in many sectors of sciences and engineering such as physics, chemistry, biology, control, electrical and mechanical engineering [6, 9, 12]. Among these, an inverse source problem (ISP) for the advection equation plays an important role in meteorology and physical oceanography. The advection equation often describes the transport of some property of the atmosphere or ocean, such as heat, humidity or salinity. Because of its applications, the ISP for the advection equation have attracted much attention of mathematicians such as Abdellatif E. B. and his colleague [1], Adel H. [3], Jose A. B. and his colleague [7], Abeer A. and his colleagues [2], so on. In particular, ISP for a space fractional advection dispersion equation on a finite domain was studied in [2]. The propagation of contaminants in the soil, oil flow in porous media, groundwater flow and turbulence can be modeled by the following space fractional advection equation

$$\begin{aligned} u_t(x,t) &= -\kappa u_x(x,t) + dD_\theta^\alpha u(x,t) + r(x),\ 0<x<L,\ t>0,\\ u(0,t) &= u(L,t) = 0,\ \ t>0,\\ u(x,T) &= g(x),\ \ 0<x<L, \end{aligned} \tag{1.1}$$

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where $u(x,t)$ is the contaminant density, κ is the average velocity, *d* is the dispersion coefficient, *r* is the source term, and α is the fractional order for the space derivative with $1<\alpha\le 2$. D_θ^α is the Riesz-Feller fractional derivative of order α, defined by [17]. Nevertheless, the authors only applied the Tikhonov regularization to construct the approximate source in numerical experiment, they did not prove the convergence of this approximate source in theory. Recently, the authors in [7] considered the problem of recovering the source function of the following advection diffusion equation

$$\begin{aligned} u_t(x,t) &= \Delta u(x,t) + \kappa u_x(x,t) + f(x),\ x\in\mathbb{R}^d,\ t>0,\\ u(x,0) &= 0,\ \ x\in\mathbb{R}^d,\\ u(x,T) &= g(x),\ \ x\in\mathbb{R}^d, \end{aligned} \tag{1.2}$$

where κ is the constant velocity of the medium and $f(x)$ is the contaminant source intensity. However, the authors only proposed the existence and uniqueness of solution, the regularization was not given.

Through the analyses of the previous results on the IPS for the advection equation, we recognize that the problem has only considered with the space dependent source. In reality, the data *g* is obtained via physical measurements, which are always established at a discrete set of points and contain errors. Beside that, these errors may be generated from controllable origins or uncontrollable origins. In the first case, the error is often deterministic, considered all metioned above articles. On the contrary, if these errors may be generated from uncontrollable origins as rain, wind, humidity, etc, then the model is random. Because of the intractable calculation of the random noise, approaches used for the deterministic cases cannot be applied directly to the random case. In practical application, the data *g* are the results of experimental measurements and it cannot be observed without errors. Therefore, in general, let $t_k=\pi\dfrac{2k-1}{2n}$ with $k=\overline{1,n}$ be grid points in $(0,\pi)$, we consider the following nonparametric regression model of data

$$g_k = g(t_k) = g(t_k) + \sigma_k\varepsilon_k,\quad k=\overline{1,n}, \tag{1.3}$$

where ε_k are unknown independent random errors. Assume that $\varepsilon_k \sim \mathcal{N}(0,1)$, and σ_k are unknown positive constants, bounded by a positive $\sigma_{\max}$, i.e., $0<\sigma_k<\sigma_{\max}$ for all $k=\overline{1,n}$. By the above reasons, we consider the problem of recovering the source in the following problem

$$\begin{aligned} &u_t+\kappa u_x = F(x,t), && x>0,\ t\in\Omega=(0,\pi),\\ &u(x,0)=0, && x>0,\\ &u(0,t)=0, && t\in(0,\pi),\\ &u(x,t)\big|_{x\to\infty}\ \text{bounded}, && t\in(0,\pi),\\ &u(1,t)=g(t), && t\in(0,\pi). \end{aligned} \tag{1.4}$$

where the source function $F(x,t)$ can be split into a product $R(x)f(t)$, and $R(x)$ is defined later. The aim of the above

problem is to identify the source term f on the basis of knowledge of the concentration of pollution u at $x=1$. We note that the perturbation of the final data g is defined by (1.3). To the best of our knowledge, there does not exist any results on ISP for the problem (1.4) with random noise in the literature. To construct the regularized heat source for the problem (1.4) - (1.3), we apply the trigonometric method in nonparametric regression associated with the truncated expansion method (refer in [11]). In addition, the convergence rate of the regularized heat source is also obtained in Section 3. Finally, we propose a numerical test to demonstrate the applicability of our method.

The rest of the paper is divided into three parts. Section 2 is devoted to establish-ment of the estimator $f_{n,N}$ for the exact source f. In Section 3, the convergence rate is obtained under a priori bound assumption of the exact source f. A numerical experiment is given to illustrate the effectiveness of our method in Section 4. Eventually, Section 5 ends this paper with a brief conclusion.

2. Discretization form of Fourier sin coefficients

Firstly, we introduce the spaces $L^2(\Omega)$ and $H^\gamma(\Omega)$ by the following propositions

Proposition 2.1 *Let*

$$L^2(\Omega) := \left\{ g : \Omega \to \mathbb{R} : \ g \text{ is Lebesgue measurable and } \int_0^\pi g^2(t)\,dt < \infty \right\},$$

with the inner product

$$\langle g_1, g_2 \rangle = \int_0^\pi g_1(t)\, g_2(t)\, dt,$$

and the norm

$$\|g\| = \left(\int_0^\pi g^2(t)\, dt \right)^{\frac{1}{2}}.$$

Proposition 2.2 *For* $p = 1, 2, \ldots,$ *we put* $\phi_p(t) = \sqrt{\frac{2}{\pi}} \sin(pt)$. *As known, the system* $\{\phi_p\}$ *is completely orthonormal. Therefore, we have the Fourier sin expansion of* $g(t)$ *as follows*

$$g(t) = \sum_{p=1}^{\infty} g_p \phi_p(t),$$

where $g_p = \langle g, \phi_p \rangle$. *The Sobolev space* $H^\gamma(\Omega)$ *of order* γ *can be naturally defined in terms of Fourier series whose coefficients decay rapidly; namely,*

$$H^\gamma(\Omega) := \left\{ g \in L^2(\Omega) : \sum_{p=1}^{\infty} p^{2\gamma} g_p^2 < \infty \right\},$$

equipped with the norm

$$\|g\|_{H^\gamma(\Omega)} = \sqrt{\sum_{p=1}^{\infty} p^{2\gamma} g_p^2}.$$

Then, by an elementary calculation the problem (1.4), we can find out the exact heat source. In particular, we get the Fourier sin expansion of $u(x,t)$

$$u(x,t) = \sum_{p=1}^{\infty} u_p(x)\phi_p(t),$$

Where $u_p(x) = \langle u(x,\cdot), \phi_p \rangle$. Substituting the above expansion into the main equation of the problem (1.4), we obtain

$$pu_p(x) + \kappa \frac{\partial}{\partial x} u_p(x) = R(x) f_p,\ u_p(0) = 0,$$

in which $f_p = \langle f, \phi_p \rangle$. By elementary calculations, we get

$$u_p(x) = \frac{1}{\kappa} \int_0^x e^{\frac{p}{\kappa}(z-x)} R(z) f_p dz. \tag{2.1}$$

Substituting $u_p(1) = \langle u(1,\cdot), \phi_p \rangle = g_p$ into (2.1) leads to

$$f_p = \frac{\kappa g_p}{\int_0^1 e^{\frac{p}{\kappa}(z-1)} R(z) dz}, \tag{2.2}$$

where $g_p = \langle g, \phi_p \rangle$. Hence, we have a Fourier sin expansion of the exact heat source f defined by

$$f(t) = \sum_{p=1}^{\infty} \frac{\kappa}{\int_0^1 e^{\frac{p}{\kappa}(z-1)} R(z) dz} g_p \phi_p(t). \tag{2.3}$$

Then, discretization form of Fourier sin coefficients g_p will be derived from the following theorem.

Lemma 2.1 ([4], page 144) *Let* $p = \overline{1, n-1}$, *and* $q = 1, 2, \ldots$, *with* $t_k = \pi \frac{2k-1}{2n}$ *and* $\phi_p(t) = \sqrt{\frac{2}{\pi}} \sin(pt)$, *then we have*

$$s_{p,q}=\sum_{k=1}^{n}\phi_p(t_k)\phi_q(t_k)=\begin{cases}\dfrac{n}{\pi}, & q\pm p=2ln\ \ (l\ even),\\ -\dfrac{n}{\pi}, & q\pm p=2ln\ \ (l\ odd),\\ 0, & otherwise.\end{cases} \tag{2.4}$$

If $q=\overline{1,n-1}$, *then*

$$s_{p,q}=\begin{cases}\dfrac{n}{\pi}, & p=q,\\ 0, & p\neq q,\end{cases} \tag{2.5}$$

and

$$\sum_{k=1}^{n}\phi_p(t_k)=\begin{cases}0, & p\neq 2ln,\\ (-1)^l\, n\sqrt{\dfrac{2}{\pi}}, & p=2ln.\end{cases}$$

Lemma 2.2 *Let* $p,n\in\mathbb{N}$ *such that* $p=\overline{1,n-1}$. *Assume that* g *is piecewise* C^1 *on* Ω. *Then we get*

$$g_p=\frac{\pi}{n}\sum_{k=1}^{n}g(t_k)\phi_p(t_k)-H_{n,p},$$

where

$$H_{n,p}=\sum_{l=1}^{\infty}(-1)^l\left[\langle g,\phi_{p+2ln}\rangle+\langle g,\phi_{-p+2ln}\rangle\right].$$

Proof. Applying the complete orthonormal basis $\{\phi_q\}_{q=1}^{\infty}$, the expansion of g can be written as follows

$$g(t_k)=\sum_{q=1}^{\infty}g_q\phi_q(t_k),$$

where $g_q=\langle g,\phi_q\rangle$. From Lemma 2.1, we get

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}g(t_k)\phi_p(t_k)&=\frac{1}{n}\sum_{k=1}^{n}\left(\sum_{q=1}^{\infty}g_q\phi_q(t_k)\right)\phi_p(t_k)\\&=\frac{1}{n}\sum_{k=1}^{n}\left(\sum_{q=1}^{n}g_q\phi_q(t_k)\right)\phi_p(t_k)+\frac{1}{n}\sum_{k=1}^{n}\left(\sum_{q=n+1}^{\infty}g_q\phi_q(t_k)\right)\phi_p(t_k)\\&=\frac{1}{n}\sum_{q=1}^{n}g_q\sum_{k=1}^{n}\phi_q(t_k)\phi_p(t_k)+\frac{1}{n}\sum_{q=n+1}^{\infty}g_q\sum_{k=1}^{n}\phi_q(t_k)\phi_p(t_k)\\&=\frac{1}{n}\frac{n}{\pi}g_p+\frac{1}{n}\frac{n}{\pi}H_{n,p}=\frac{1}{\pi}\left(g_p+H_{n,p}\right).\end{aligned}$$

The proof of Lemma 2.2 is complete. ■

From the result of Lemma 2.2, we can obtain a data-explicit form for the exact heat source f.

Theorem 2.3 *Let $N \in \mathbb{N}$ such that $0 < N \le n$. Assume that the function g is fulfilled Lemma 2.2. Then the exact heat source f is defined by*

$$f(t) = \sum_{p=1}^{N} \frac{\kappa}{\int_0^1 e^{\frac{p}{\kappa}(z-1)} R(z)\,dz} \left(\frac{\pi}{n} \sum_{k=1}^{n} g(t_k)\phi_p(t_k) - H_{n,p} \right) \phi_p(t) + \sum_{p=N+1}^{\infty} \frac{\kappa}{\int_0^1 e^{\frac{p}{\kappa}(z-1)} R(z)\,dz} g_p \phi_p(t). \tag{2.6}$$

Here we call N the regularized parameter which depends on n. The notation $H_{n,p}$ is defined in Lemma 2.2.

Proof. From Lemma 2.2 and the formula (2.3), we obtain

$$\begin{aligned} f(t) &= \sum_{p=1}^{\infty} \frac{\kappa}{\int_0^1 e^{\frac{p}{\kappa}(z-1)} R(z)\,dz} g_p \phi_p(t) \\ &= \sum_{p=1}^{N} \frac{\kappa}{\int_0^1 e^{\frac{p}{\kappa}(z-1)} R(z)\,dz} g_p \phi_p(t) + \sum_{p=N+1}^{\infty} \frac{\kappa}{\int_0^1 e^{\frac{p}{\kappa}(z-1)} R(z)\,dz} g_p \phi_p(t) \\ &= \sum_{p=1}^{N} \frac{\kappa}{\int_0^1 e^{\frac{p}{\kappa}(z-1)} R(z)\,dz} \left(\frac{\pi}{n} \sum_{k=1}^{n} g(t_k)\phi_p(t_k) - H_{n,p} \right) \phi_p(t) \\ &\quad + \sum_{p=N+1}^{\infty} \frac{\kappa}{\int_0^1 e^{\frac{p}{\kappa}(z-1)} R(z)\,dz} g_p \phi_p(t). \end{aligned}$$

The proof of Theorem 2.3 is complete. Estimators and Convergence results ■

Now two important lemmas are given as follows

Lemma 3.1 *Let $R:[0,T] \to \mathbb{R}$ be a positive continuous function such that $\inf_{t\in[0,T]} R(t) = R_0 > 0$. Set $\|R\|_\infty = \sup_{t\in[0,T]} R(t)$. Then we have for all $p \in \mathbb{N}^*$*

$$\frac{\kappa R_0}{p}\left(1 - e^{-\frac{1}{\kappa}}\right) \le \int_0^1 e^{\frac{p}{\kappa}(z-1)} R(z)\,dz \le \frac{\kappa \|R\|_\infty}{p}.$$

Proof. Firstly, we have

$$\int_0^1 e^{\frac{p}{\kappa}(z-1)}\,dz = \frac{\kappa}{p}\left(1 - e^{-\frac{p}{\kappa}}\right).$$

This implies that

$$\int_0^1 e^{\frac{p}{\kappa}(z-1)} R(z)\,dz \le \sup_{t\in[0,T]} R(t) \int_0^1 e^{\frac{p}{\kappa}(z-1)}\,dz \le \frac{\kappa \|R\|_\infty}{p},$$

and

$$\int_0^1 e^{\frac{p}{\kappa}(z-1)} R(z)\,dz \ge \inf_{t\in[0,T]} R(t) \int_0^1 e^{\frac{p}{\kappa}(z-1)}\,dz = \frac{\kappa R_0}{p}\left(1-e^{-\frac{p}{\kappa}}\right) \ge \frac{\kappa R_0}{p}\left(1-e^{-\frac{1}{\kappa}}\right).$$

The proof of Lemma 3.1 is complete. ■

Then we can obtain the following conditional stability.

Theorem 3.1 *If there exists* $M \ge 0$ *such that* $\|f\|_{H^\gamma(0,\pi)} \le M$, *for* $\gamma > 0$, *then*

$$\|f\| \le \left[\frac{1}{R_0\left(1-e^{-\frac{1}{\kappa}}\right)}\right]^{\frac{2\gamma}{\gamma+2}} M^{\frac{4}{\gamma+2}} \|g\|^{\frac{2\gamma}{\gamma+2}}.$$

Proof. Using Hölder's inequality, we first have

$$\|f\|^2 = \sum_{p=1}^{\infty} \kappa^2 \left(\int_0^1 e^{\frac{p}{\kappa}(z-1)} R(z)\,dz\right)^{-2} g_p^{\frac{4}{\gamma+2}} g_p^{\frac{2\gamma}{\gamma+2}}$$

$$\le \sum_{p=1}^{\infty} \left[\kappa^{\gamma+2}\left(\int_0^1 e^{\frac{p}{\kappa}(z-1)} R(z)\,dz\right)^{-(\gamma+2)} g_p^2\right]^{\frac{2}{\gamma+2}} \left(\sum_{p=1}^{\infty} g_p^2\right)^{\frac{\gamma}{\gamma+2}}.$$

From (2.2), the inequality will be obtained continuously.

$$\|f\|^2 \le \sum_{p=1}^{\infty} \left[\kappa^{\gamma}\left(\int_0^1 e^{\frac{p}{\kappa}(z-1)} R(z)\,dz\right)^{-\gamma} f_p^2\right]^{\frac{2}{\gamma+2}} \|g\|^{\frac{2\gamma}{\gamma+2}}.$$

From Lemma 3.1, we get

$$\|f\|^2 \le \sum_{p=1}^{\infty} \left[\kappa^{\gamma}\left(\int_0^1 e^{\frac{p}{\kappa}(z-1)} R(z)\,dz\right)^{-\gamma} f_p^2\right]^{\frac{2}{\gamma+2}} \|g\|^{\frac{2\gamma}{\gamma+2}}$$

$$\le \sum_{p=1}^{\infty} \left[\kappa^{\gamma}\left(\frac{\kappa R_0\left(1-e^{-\frac{1}{\kappa}}\right)}{p}\right)^{-\gamma} f_p^2\right]^{\frac{2}{\gamma+2}} \|g\|^{\frac{2\gamma}{\gamma+2}}$$

$$\le \left[\frac{1}{R_0\left(1-e^{-\frac{1}{\kappa}}\right)}\right]^{\frac{2\gamma}{\gamma+2}} \left(\sum_{p=1}^{\infty} p^{2\gamma} f_p^2\right)^{\frac{2}{\gamma+2}} \|g\|^{\frac{2\gamma}{\gamma+2}}$$

$$= \left[\frac{1}{R_0\left(1-e^{-\frac{1}{\kappa}}\right)}\right]^{\frac{2\gamma}{\gamma+2}} \|f\|_{H^\gamma(\Omega)}^{\frac{4}{\gamma+2}} \|g\|^{\frac{2\gamma}{\gamma+2}}.$$

The proof of Lemma 3.1 is complete. ■

Lemma 3.2 *Let* $p,n\in\mathbb{N}$ *such that* $p=\overline{1,n-1}$. *Assume that the functions* $g,H_{n,p},f$ *are fulfilled in Lemma 2.2 and Lemma 3.1, we have the following inequality*

$$\left|H_{n,p}\right|\le 2K_\gamma\frac{\left\|R\right\|_\infty\left\|f\right\|_{H^\gamma(\Omega)}}{n^{1+\gamma}},$$

where the function f *is the exact heat source and*

$$K_\gamma:=\sum_{l=1}^{\infty}\frac{1}{l^{1+\gamma}}.$$

Proof. From (2.2), we have

$$g_p=\int_0^1 e^{\frac{p}{\kappa}(z-1)}R(z)\,dz\,\frac{f_p}{\kappa},$$

Applying the convergence of the following series

$$K_\gamma:=\sum_{l=1}^{\infty}\frac{1}{l^{1+\gamma}}<+\infty,$$

we obtain

$$\begin{aligned}
\left|H_{n,p}\right|&=\left|\sum_{l=1}^{\infty}(-1)^l\left[\left\langle g,\phi_{p+2ln}\right\rangle+\left\langle g,\phi_{-p+2ln}\right\rangle\right]\right|\\
&\le\left|\sum_{l=1}^{\infty}\left\langle g,\phi_{p+2ln}\right\rangle\right|+\left|\sum_{l=1}^{\infty}\left\langle g,\phi_{-p+2ln}\right\rangle\right|\\
&\le\sum_{l=1}^{\infty}\left|\int_0^1 e^{\frac{p+2ln}{\kappa}(z-1)}R(z)\,dz\,\frac{f_{p+2ln}}{\kappa}\right|+\sum_{l=1}^{\infty}\left|\int_0^1 e^{\frac{p+2ln}{\kappa}(z-1)}R(z)\,dz\,\frac{f_{-p+2ln}}{\kappa}\right|\\
&\le\sum_{l=1}^{\infty}\left|\frac{\left\|R\right\|_\infty}{p+2ln}\,f_{p+2ln}\right|+\sum_{l=1}^{\infty}\left|\frac{\left\|R\right\|_\infty}{-p+2ln}\,f_{-p+2ln}\right|\\
&\le\left\|R\right\|_\infty\left\|f\right\|_{H^\gamma(\Omega)}\left[\sum_{l=1}^{\infty}\frac{1}{\left(p+2ln\right)^{1+\gamma}}+\sum_{l=1}^{\infty}\frac{1}{\left(-p+2ln\right)^{1+\gamma}}\right]\\
&=2K_\gamma\frac{\left\|R\right\|_\infty\left\|f\right\|_{H^\gamma(\Omega)}}{n^{1+\gamma}}.
\end{aligned}$$

The proof of Lemma 3.2 is complete. ■

Next, we construct a regularized approximate heat source of the exact heat source (2.6) $f_{n,N}$, defined by

$$f_{n,N}(t)=\sum_{p=1}^{N}\frac{\kappa}{\int_0^1 e^{\frac{p}{\kappa}(z-1)}R(z)\,dz}\left(\frac{\pi}{n}\sum_{k=1}^{n}g(t_k)\phi_p(t_k)\right)\phi_p(t),\tag{3.1}$$

we recall N the regularized parameter which depends on n. Moreover, the following theorem is a main result of this paper.

Theorem 3.2 *Let $\varepsilon_k \sim \mathcal{N}(0,1)$ be independent normal random variables with $k=\overline{1,n}$ (as mentioned above). Assume that the exact heat source f is fulfilled in Lemma 3.1, and the functions f and $f_{n,N}$ defined by (2.6) and (3.1), are the exact heat source and its regularized heat source corresponding to the nonparametric regression model of data g, respectively. Then we have the following estimate*

$$\mathbb{E}\left\|f_{n,N}-f\right\|^2 \le \frac{2}{R_0^2\left(1-e^{-\frac{1}{\kappa}}\right)^2}\left(\frac{\pi^2}{n}\sigma_{\max}^2+2K_\gamma^2\frac{\|R\|_\infty^2\|f\|_{H^\gamma(\Omega)}^2}{n^{2+2\gamma}}\right)N^3+N^{-2\gamma}M^2. \tag{3.2}$$

Proof. Applying (2.6) and (3.1), we obtain

$$\begin{aligned} f_{n,N}(t)-f(t) &= \sum_{p=1}^{N}\frac{\kappa}{\int_0^1 e^{\frac{p}{\kappa}(z-1)}R(z)dz}\left(\frac{\pi}{n}\sum_{k=1}^{n}\sigma_k\varepsilon_k\phi_p(t_k)+H_{n,p}\right)\phi_p(t) \\ &\quad -\sum_{p=N+1}^{\infty}\frac{\kappa}{\int_0^1 e^{\frac{p}{\kappa}(z-1)}R(z)dz}g_p\phi_p(t). \end{aligned} \tag{3.3}$$

By Parseval's identity, we have

$$\begin{aligned} \left\|f_{n,N}-f\right\|^2 &= \sum_{p=1}^{N}\frac{\kappa^2}{\left(\int_0^1 e^{\frac{p}{\kappa}(z-1)}R(z)dz\right)^2}\left(\frac{\pi}{n}\sum_{k=1}^{n}\sigma_k\varepsilon_k\phi_p(t_k)+H_{n,p}\right)^2 \\ &\quad +\sum_{p=N+1}^{\infty}\frac{\kappa^2}{\left(\int_0^1 e^{\frac{p}{\kappa}(z-1)}R(z)dz\right)^2}g_p^2. \end{aligned}$$

Since the noises ε_k are mutually independent, we obtain $\mathbb{E}(\varepsilon_i\varepsilon_k)=0$, with $i\ne k$. Hence

$$\begin{aligned} \mathbb{E}\left\|f_{n,N}-f\right\|^2 &\le \underbrace{2\sum_{p=1}^{N}\frac{\kappa^2}{\left(\int_0^1 e^{\frac{p}{\kappa}(z-1)}R(z)dz\right)^2}\left[\frac{\pi^2}{n^2}\sum_{k=1}^{n}\sigma_k^2\mathbb{E}\varepsilon_k^2+H_{n,p}^2\right]}_{=:B_1} \\ &\quad +\underbrace{\sum_{p=N+1}^{\infty}\frac{\kappa^2}{\left(\int_0^1 e^{\frac{p}{\kappa}(z-1)}R(z)dz\right)^2}g_p^2}_{=:B_2}. \end{aligned} \tag{3.4}$$

Step 1. Estimate B_1.

From Lemma 3.1 and Lemma 3.2, we get

$$B_1 \le \left[2\frac{\pi^2}{n}\sigma_{\max}^2 + 4K_\gamma^2 \frac{\|R\|_\infty^2 \|f\|_{H^\gamma(\Omega)}^2}{n^{2+2\gamma}} \right] \sum_{p=1}^{N} \frac{\kappa^2}{\left(\int_0^1 e^{\frac{p}{\kappa}(z-1)} R(z)\,dz \right)^2}$$

$$\le 2\kappa^2 \left(\frac{\pi^2}{n}\sigma_{\max}^2 + 2K_\gamma^2 \frac{\|R\|_\infty^2 \|f\|_{H^\gamma(\Omega)}^2}{n^{2+2\gamma}} \right) \sum_{p=1}^{N} \frac{p^2}{\kappa^2 R_0^2 \left(1 - e^{-\frac{1}{\kappa}} \right)^2} \tag{3.5}$$

$$\le \frac{2}{R_0^2 \left(1 - e^{-\frac{1}{\kappa}} \right)^2} \left(\frac{\pi^2}{n}\sigma_{\max}^2 + 2K_\gamma^2 \frac{\|R\|_\infty^2 \|f\|_{H^\gamma(\Omega)}^2}{n^{2+2\gamma}} \right) N^3. \tag{3.6}$$

Step 2. Estimate B_2.

From (2.2), we can rewrite B_2 as follows

$$B_2 = \sum_{p=N+1}^{\infty} p^{-2\gamma} p^{2\gamma} f_p^2 \le N^{-2\gamma} \sum_{p=1}^{\infty} p^{2\gamma} f_p^2 \le N^{-2\gamma} M^2. \tag{3.7}$$

Combining (3.4), (3.6) and (3.7), we deduce that

$$\mathbb{E}\left\| f_{n,N} - f \right\|^2 \le \frac{2}{R_0^2 \left(1 - e^{-\frac{1}{\kappa}} \right)^2} \left(\frac{\pi^2}{n}\sigma_{\max}^2 + 2K_\gamma^2 \frac{\|R\|_\infty^2 \|f\|_{H^\gamma(\Omega)}^2}{n^{2+2\gamma}} \right) N^3 + N^{-2\gamma} M^2.$$

This completes the proof of Theorem 3.2. ■

Remark 3.3 *Let* $N := N(n)$ *such that* $0 < N < n$ *and*

$$\lim_{n \to +\infty} \frac{N^3}{n} = 0,$$

then

$$\mathbb{E}\left\| f_{n,N} - f \right\|^2 \text{ is of order } \max\left(\frac{N^3}{n}, N^{-2\gamma} \right). \tag{3.8}$$

Remark 3.4 *If* $N = n^{\frac{1}{6}}$, *from (3.8), in the case* $0 < \gamma \le \frac{3}{2}$, *we infer that*

$$\mathbb{E}\left\| f_{n,N} - f \right\|^2 \text{ is of order } \frac{1}{n^{\frac{\gamma}{3}}},$$

and in the case $\gamma > \frac{3}{2}$, *we conclude that*

$$\mathbb{E}\left\| f_{n,N} - f \right\|^2 \text{ is of order } \left(\frac{1}{n}\right)^{\frac{1}{2}}.$$

Remark 3.5 *By choosing* $N = n^{\frac{1}{3+2\gamma}}$ *and (3.8), we get*

$$\mathbb{E}\left\| f_{n,N} - f \right\|^2 \text{ is of order } \left(\frac{1}{n}\right)^{\frac{2\gamma}{3+2\gamma}}. \tag{3.9}$$

3. Numerical experiment

In this section, we are going to show a numerical experiment to illustrate the efficiency of our scheme. It will be observed by comparing the results between numerical and exact sources. We shall choose given functions in such a way that they lead to a given exact solution. On the other hand, numerical results with many 2-D graphs shall be discussed in this section. Now we determine the heat source of the following problem

$$\begin{aligned} &u_t + u_x = R(x) f(t), \quad x > 0,\ t \in \Omega = (0,\pi), \\ &u(x,0) = 0, \qquad x > 0, \\ &u(0,t) = 0, \qquad t \in (0,\pi), \\ &u(x,t)\big|_{x\to\infty} \text{ bounded}, \qquad t \in (0,\pi), \\ &u(1,t) = g(t), \qquad t \in (0,\pi). \end{aligned} \tag{4.1}$$

in which

$$g(t) = \sqrt{\frac{2}{\pi}} \sin(t), \ \ t \in (0,\pi),$$

$$R(x) = e^{-x}, \ \ x > 0.$$

It implies that the exact source f of the problem (4.1) is of the following form

$$f(t) = \sqrt{\frac{2}{\pi}} e \sin(t). \tag{4.2}$$

Let the sequence $\{g_k\}_{k=1}^n$ represent samples from the function $g(t)$ on an equidistant grid. Choosing $\sigma_k^2 = \sigma^2 = 10^{-i}$, $i = \overline{1,3}$, we have the following regression model

$$g_k = g(t_k) = g(t_k) + \sigma\varepsilon_k, \ \ t_k = \pi\frac{2k-1}{2n}, \ \ k = \overline{1,n},$$

where $\varepsilon_k \sim \mathcal{N}(0,1)$.

From (3.1), we have the regularized heat source of the exact heat source (4.2) as follows

$$f_{n,N}(t)=\sum_{p=1}^{N}\frac{1}{\int_0^1 e^{pz-p-z}dz}\left[\frac{\pi}{n}\sum_{k=1}^{n}g(t_k)\phi_p(t_k)\right]\phi_p(t). \tag{4.3}$$

To calculate the error between the exact solution and the estimator, we use the root mean squared error (RMSE) as follows

$$RMSE\left(f_{n,N},f\right)=\sqrt{\frac{1}{n}\sum_{k=1}^{n}\left[f_{n,N}(t_k)-f(t_k)\right]^2}.$$

Then Table 1 shows the error estimate in case $\sigma^2=10^{-i}$, $i=\overline{1,3}$, $n=100$ by two choices of the regularized parameter $N=n^{\frac{1}{6}}$ and $N=n^{\frac{1}{3+2\beta}}$, $\beta=\frac{1}{4}$. The results of this experiment have demonstrated numerically the effectiveness of the estimator.

Table 1: The error estimate between the exact source and the regularized source in case $n=100$

	$RMSE\left(f_{n,N},f\right)$		
σ^2	10^{-1}	10^{-2}	10^{-3}
$N=n^{\frac{1}{6}}$	0.0298	1.8184×10^{-3}	6.5020×10^{-4}
$N=n^{\frac{1}{3+2\beta}}$	0.0616	5.6883×10^{-3}	2.8419×10^{-4}

From Table 1, we see that the errors (in three cases of the variance $\sigma_k^2=\sigma$) are decreased when the regularized parameter N is increased. Finally, Figure 1 - 3 show comparison between the exact solution and its regularized solution. It again shows that the regularized source was strong oscillated around the exact source.

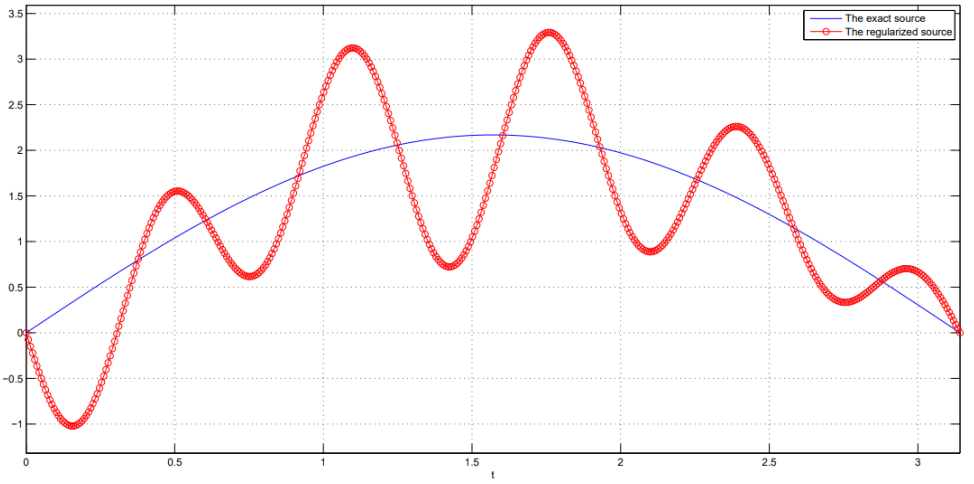


Figure 1: The graph of the exact source and the regularized source in case $\sigma^2=10^{-1}$ and $N=n^{\frac{1}{6}}$.

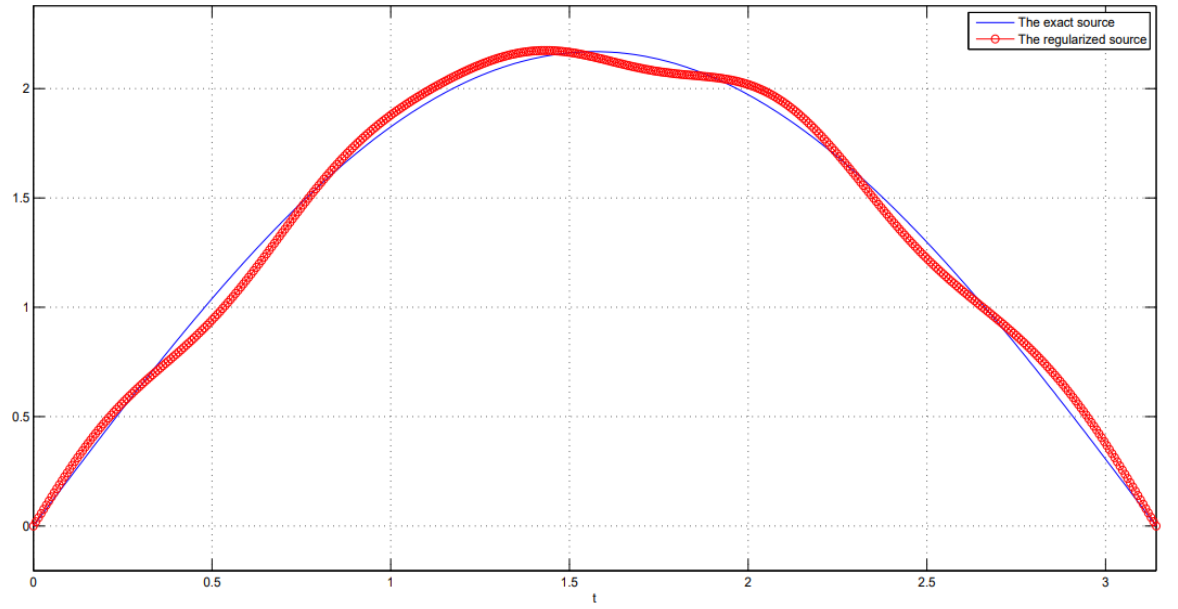


Figure 2: The graph of the exact source and the regularized source in case $\sigma^2 = 10^{-2}$ and $N = n^{\frac{1}{6}}$.

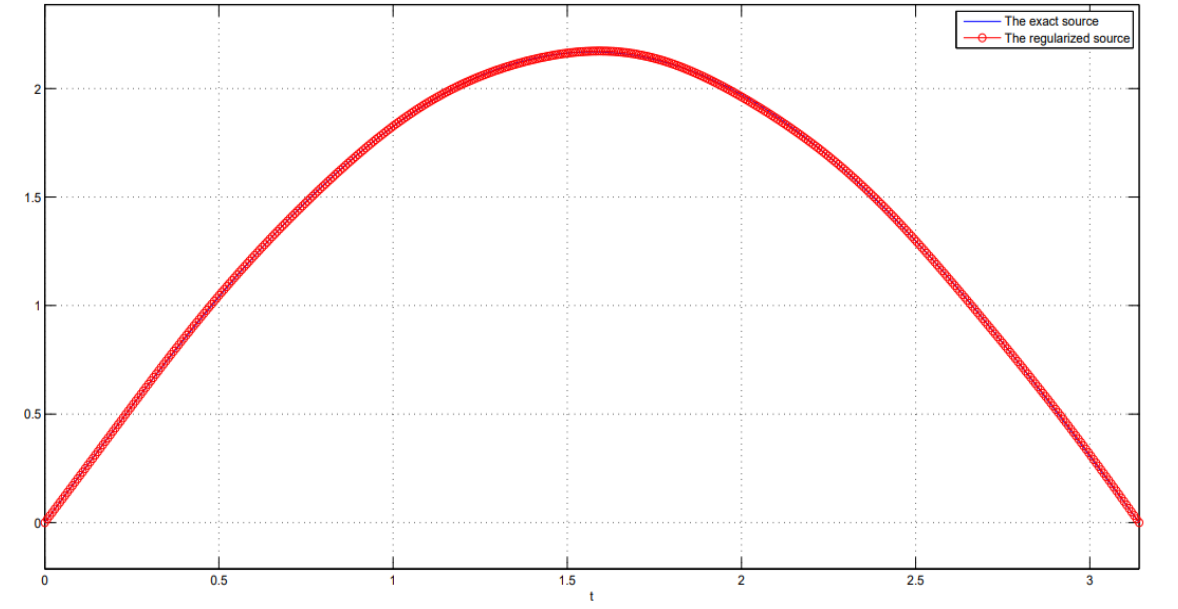


Figure 3: The graph of the exact source and the regularized source in case $\sigma^2 = 10^{-3}$ and $N = n^{\frac{1}{6}}$.

4. Conclusion

In this study, we solved problem (1.4) - (1.3) to recover time-dependent function of the unknown source in the statistical advection equation (i.e., inhomogeneous source) by suggesting the trigonometric method in nonparametric regression associated with the

truncated expansion method to approximate stably the exact source *f*. Moreover, the estimate of bias of the discretization is given explicitly in Section 3. Finally, we illustrate the theoretical part by a numerical experiment in Section 4.

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