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A BACKWARD PROBLEM FOR A FRACTIONAL PSEUDO - PARABOLIC EQUATION WITH A NONHOMOGENOUS SOURCE TERM

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Abstract

In this paper, we deal with the backward problem for a fractional pseudo - parabolic equation with a nonhomogenous source term. A fractional pseudo - parabolic equation is applied to describe a flow of a fluid caused by a force affect on the solid boundary. In particular, we prove that the problem is ill-posedness in the sense of Hadamard. To regularize the problem, we propose two modified versions of the so-called optimal filtering method of Seidman. According to different *a priori* assumptions on the regularity of the exact solution, we obtain some sharp optimal estimates between the exact solution and its approximated solution of Hölder – logarithmic type.

Keywords: *Backward problem, Fourier transform, Pseudo-parabolic equation, Ill-posed problems*

1. Introduction

Recently, partial differential equations (PDEs) are important play a role of mathematical model of a wide class of important anomalous diffusion and mechanical fields. At the end of the 17th century, fractional derivatives appeared and flourished throughout the centuries because it has many improvements over the classical derivative when considering phenomena with memory effects [1]. On that basis, fractional differential equations (FDEs) were born and contributed many applications to describe some natural events. They have many applications in engineering which can be used to describe a wide variety of phenomena such as sound, heat, diffusion, electrodynamics, fluid dynamics, elasticity, and gravitation [1-8].

There are many different types of partial differential equations, in the current paper, we are interested in the time-fractional pseudo-parabolic equation. In the theory for incompressible fluids by Coleman and Walter [10], the classical pseudo-parabolic equation can be seen as the restricted form of the constitutive equation of their theory. According to Ting [11], the equation:

$$a\partial_t w(x,t) - b\Delta w(x,t) - c\partial_t w(x,t) = 0, \quad (1)$$

where x and t are variables in some certain domains and a, b, and c are parameters depending on some specific phenomena, has many important applications in fluid mechanics. For example, an equation with some reasonable boundary conditions can be used to model a flow of a fluid caused by a force affect on the solid boundary. We can also list some other notable applications of this type of PDEs as follows.

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- The hydrostatic excess pressure within a portion of clay during consolidation (see Talor [12]);
- The leakage of liquid through cracks in rocks or materials (see Xu and Su [13]);
- The aggregation of populations (see Padron [14]).

In this paper, we study a backward problem for the time-fractional pseudo - parabolic equation with a nonhomogenous source term:

$$\begin{cases} u_t + a\mathcal{A}_{x,n}u_t - \mathcal{A}^v_{x,n}u = f(x,t), & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,T) = \varphi(x), & \text{in } \mathbb{R}^n, \end{cases} \quad (1.1)$$

Here $x = (x_1, x_2, \ldots, x_n)$ is a n -dimensional variable, a is positive number, $f : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ is the nonhomogenous and $T > 0, n \in N^* = \{1,2,\ldots\}$. In the above problem, the constant a plays the role as a characteristic of the fissured rock, a reduction in block dimension implies a decrease of a and its increase in the degree of fissuring. Moreover, the constant a also has other applications such as aggregation of populations, the unidirectional propagation of nonlinear, long waves, dispersive and so on. The problem (1.1) is known as a fractional pseudo - parabolic problem with a nonhomogenous source term, which aims to obtain the history of a heat process using values at the final time T.

Our results on the problem (1.1) are listed as follows: Our paper is the first result on backward problem for pseudo-parabolic equation with Caputo derivative. Here, we prove the ill-posedness of this problem (in the sense of Hadamard). To regularize the problem, we propose two modified versions of the so-called optimal filtering method of Seidman [18]. According to different *a priori* assumptions on the regularity of the exact solution, we obtain some sharp optimal estimates.

As we known, the well-posedness and regularity properties for the solution of PDEs for $L^p\left(p \neq 2\right)$ setting is a difficult problem. To overcome these difficulties when we consider $L^p\left(p \neq 2\right)$ setting, we have to use optimal filtering method of Seidman.

Condition 1. Throughout this paper, we assume that the symbol ϕ_n satisfies the following condition $(\mathcal{C})$ for any $\alpha > 0, v > 1$ and $\bar{\alpha} = \alpha\left(v - 1\right)$ there exist two constants $c_1, c_2 > 0$ such that:

$$(\mathcal{C}): c_1 \mid \xi \mid^{\bar{\alpha}} \leq \mathrm{Re}\left(\frac{\phi_n^v(\xi)}{1 + a\phi_n(\xi)}\right) \leq c_2 \mid \xi \mid^{\bar{\alpha}}, \forall \xi \in \mathbb{R}^n.$$

Here and the remaining part of this article, $\mid \xi \mid$ denotes the Euclidian norm of $\xi \in \mathbb{R}^n$ and $\mathrm{Re}\left(z\right)$ denotes the real of the complex number z.

This paper is organized as follows. In Section 2, we introduce some basic definitions and the ill-posedness of the problem (1.1). In Section 3, the main result of this paper the following is **Theorem 3.1.** Finally, Section 4 we present the proof of **Theorem 3.1.**

2. The ill-posedness of the problem (1.1)

We will see in **Section 2** that the problem (1.1) ill-posed in the sense of Hadamard. That is, the solution, if it exists, does not depend continuously on the data and any small perturbation in the given data φ may cause a large change to the solution. The ill-posedness can be easily seen by considering the problem in the Fourier domain.

Firstly, we recall some basic definition.

Definition 2.1. For the sake of convenience, we denote

$$L^2\left(\mathbb{R}^n\right) := \left\{ f : \mathbb{R}^n \to \mathbb{R} \text{ measured Lebesgue}, \int\limits_{\mathbb{R}^n} \left| f\left(x\right) \right|^2 \mathrm{d}x < \infty \right\},$$

with norm $\left\| f \right\| = \left(\int\limits_{\mathbb{R}^n} \left| f\left(x\right) \right|^2 \mathrm{d}x \right)^{\frac{1}{2}}$.

Definition 2.2. For $q > 0,$ we recall the Sobolev space $H^q\left(\mathbb{R}^n\right)$ as follows

$$H^q\left(\mathbb{R}^n\right) := \left\{ f \in L^2\left(\mathbb{R}^n\right), \int\limits_{\mathbb{R}^n} \left(1 + |\, \xi \,|^2\right)^q \left| f\left(x\right) \right|^2 \mathrm{d}x < \infty \right\},$$

with norm $\left\| f \right\|_{H^q\left(\mathbb{R}^n\right)} := \left(\int_{\mathbb{R}^n} \left(1 + |\, \xi \,|^2\right)^q |\, \hat{f}(\xi) \,|^2 \mathrm{d}\xi \right)^{\frac{1}{2}}$. For brevity's sake, we define $\langle \cdot \rangle := \left(1 + |\cdot|^2\right)^{\frac{1}{2}}$.

Then, we restate theory of the Fourier transform in the following definition.

Definition 2.3. Assume that $f \in L^2\left(\mathbb{R}^n\right)$, $\mathcal{F}$ denotes the n-dimensional Fourier transform operator and is defined as

$$\mathcal{F}[f](\xi) := \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} f(x) \mathrm{e}^{-i\xi \cdot x} \mathrm{d}x = \hat{f}(\xi), \quad i^2 = -1,$$

while $\mathcal{F}^{-1}$ is the n-dimensional inverse Fourier transform operator such that

$$\mathcal{F}^{-1}[\mathcal{F}[f](\xi)](x) = f(x).$$

Here $\xi \cdot x$ is the dot product of the Cartesian coordinates of two vectors ξ and x in $\mathbb{R}^n$.

Definition 2.4. For $v > 1$, the operator $\mathcal{A}_{x,n}$ and $\mathcal{A}_{x,n}^v$ are a pseudo-differential operator and a fractional pseudo - differential operator with respect to the variable x in the n-dimensional space, respectively, defined by:

$$\mathcal{A}_{x,n} f(x) := \mathcal{F}^{-1}\left[\phi_n(\xi)\mathcal{F}[f](\xi)\right](x),$$

$$\mathcal{A}_{x,n}^v f(x) := \mathcal{F}^{-1}\left[\phi_n^v(\xi)\mathcal{F}[f](\xi)\right](x).$$

where the function $f \in L^2\left(\mathbb{R}^n\right)$ and the function $\phi_n : \mathbb{R}^n \to \mathbb{C}$ is called the symbol of the operator $\mathcal{A}_{x,n}$ and $\mathcal{A}^v_{x,n}$.

Remark. The motivation of the condition $(\mathcal{C})$ is that some fractional differential operators can be written as multipliers over the frequency space, and their growth as a function of ξ is polynomial. A simple example for such an operator is $\mathcal{A}_{x,n} = -\left(-\Delta_{x,n}\right)^{\frac{\alpha}{2}}, \alpha \in (0,2]$, where the operator $\left(-\Delta_{x,n}\right)^{\frac{\alpha}{2}}$ called the n-dimensional fractional Laplacian operator of order $\frac{\alpha}{2}$, is defined by $\left[20,\ Section\ 2.2.1\right]$.

$$\mathcal{F}\left[-\Delta^{\frac{\alpha}{2}}_{x,n} f(x)\right](\xi) := \mid \xi \mid^{\alpha}\ \mathcal{F}[f](\xi).$$

2.1. The exact solution of the problem (1.1)

Beside that, $\mathcal{A}^v_{x,n}$ with $v > 1$ is the fractional pseudo-differential operation with expect to the variable x and defined by

$$\widehat{u_t}(\xi,t) + a\phi_n(\xi)\widehat{u_t}(\xi,t) + \phi^v_n(\xi)\hat{u}(\xi,t) = \hat{f}(\xi,t).$$

By solving for the first order original differential equation, we obtain the exact ridits of the problem (1.1).

$$\hat{u}(\xi,t) = \hat{u}(\xi,0)\exp\left(\frac{-\phi^v_n(\xi)t}{1+a\phi_n(\xi)}\right) + \int_0^t \frac{1}{1+a\phi_n(\xi)}\exp\left(\frac{\phi^v_n(\xi)(s-t)}{1+a\phi_n(\xi)}\right)\hat{f}(\xi,s)\mathrm{d}s.$$

Replacing $t = T$ and $\hat{u}(\xi,T) = \hat{\varphi}(\xi)$, we have

$$\hat{\varphi}(\xi) = \hat{u}(\xi,0)\exp\left(\frac{-\phi^v_n(\xi)T}{1+a\phi_n(\xi)}\right) + \int_0^T \frac{1}{1+a\phi_n(\xi)}\exp\left(\frac{\phi^v_n(\xi)(s-T)}{1+a\phi_n(\xi)}\right)\hat{f}(\xi,s)\mathrm{d}s.$$

It follows that

$$\hat{u}(\xi,0) = \exp\left(\frac{\phi^v_n(\xi)T}{1+a\phi_n(\xi)}\right)\left[\hat{\varphi}(\xi) - \int_0^T \frac{1}{1+a\phi_n(\xi)}\exp\left(\frac{\phi^v_n(\xi)(s-T)}{1+a\phi_n(\xi)}\right)\hat{f}(\xi,s)\mathrm{d}s\right].$$

We infer that

$$\hat{u}(\xi,t) = \exp\left(\frac{\phi^v_n(\xi)\left(T-t\right)}{1+a\phi_n(\xi)}\right)\hat{\varphi}(\xi) - \int_t^T \exp\left(\frac{\phi^v_n(\xi)\left(s-t\right)}{1+a\phi_n(\xi)}\right)\hat{f}(\xi,s)\mathrm{d}s.$$

Applying the inverse Fourier transform with respect to ξ in (1.1), we can establish the relation between $u(x,t)$ and $\varphi(x)$ in frequency domain as

$$u(x,t) = A_1\left(T-t\right)\varphi(\xi) - \int_t^T A_2\left(s-t\right)f(x,s)\mathrm{d}s \qquad (2.1)$$

where

$$A_1\left(t\right)v\left(x\right) = \frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n}\exp\left(\frac{\phi_n^v(\xi)t}{1+a\phi_n(\xi)}\right)\hat{v}(\xi)\mathrm{e}^{ix\cdot\xi}\mathrm{d}\xi,$$
$$A_2\left(t\right)v\left(x\right) = \frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n}\frac{1}{1+a\phi_n(\xi)}\exp\left(\frac{\phi_n^v(\xi)t}{1+a\phi_n(\xi)}\right)\hat{v}(\xi)\mathrm{e}^{ix\cdot\xi}\mathrm{d}\xi.$$

2.2. The ill-posedness of the problem (1.1)

In this section, we present an example to show that the problem (1.1) is ill-posed in the sense of Hadamard.

Example 1. We first construct the input data $\varphi_n \in L^2\left(\mathbb{R}^n\right)$ defined by the Fourier transform as follows

$$\widehat{\varphi_n}(\xi) = \begin{cases}\hat{\varphi}(\xi), & \text{if } \mid \xi - n \mid > 1,\\ \hat{\varphi}(\xi) + \dfrac{1}{n\sqrt{2}}, & \text{if } \mid \xi - n \mid \le 1.\end{cases}$$

Using the Parseval's formula, we know that

$$\left\|\varphi_n - \varphi\right\|^2_{L^2(\mathbb{R})} = \left\|\widehat{\varphi_n} - \widehat{\varphi}\right\|^2_{L^2(\mathbb{R})} = \int\limits_{|\xi-n|\le 1}\frac{1}{2n^2}\,d\xi = \frac{1}{n^2}.$$

Consequently,

$$\lim_{n\to+\infty}\left\|\varphi_n - \varphi\right\|_{L^2(\mathbb{R})} = \lim_{n\to+\infty}\frac{1}{n} = 0 \qquad (2.2)$$

Let u and u_n be two solutions of problem (1.1) corresponding to the final values (f,φ) and (f,φ_n), that is

$$\hat{u}(\xi,t) = \exp\left(\frac{\phi_n^v(\xi)\left(T-t\right)}{1+a\phi_n(\xi)}\right)\hat{\varphi}(\xi) - \int_t^T\exp\left(\frac{\phi_n^v(\xi)\left(s-t\right)}{1+a\phi_n(\xi)}\right)\hat{f}(\xi,s)\mathrm{d}s,$$

and

$$\widehat{u_n}(\xi,t) = \exp\left(\frac{\phi_n^v(\xi)\left(T-t\right)}{1+a\phi_n(\xi)}\right)\widehat{\varphi_n}(\xi) - \int_t^T\exp\left(\frac{\phi_n^v(\xi)\left(s-t\right)}{1+a\phi_n(\xi)}\right)\hat{f}(\xi,s)\mathrm{d}s.$$

By simple computation, we have

$$| \widehat{u_n}(\xi,t) - \hat{u}(\xi,t) | = \left| \exp\left(\frac{\phi_n^v(\xi)\big(T-t\big)}{1+a\phi_n(\xi)} \right) \Big(\widehat{\varphi_n}(\xi) - \widehat{\varphi}\ (\xi) \Big) \right|.$$

Because of (C) and $| e^z | = e^{\mathrm{Re}z}$, we have that

$$\left| \exp\left(\frac{\phi_n^v(\xi)(T-t)}{1+a\phi_n(\xi)} \right) \right| = \exp\left(Re\left(\frac{\phi_n^v(\xi)(T-t)}{1+a\phi_n(\xi)} \right) \right) \geq \exp\Big(c_1 \mid \xi \mid^{\bar{\alpha}} (T-t) \Big).$$

It follows from this and $| \xi | \geq | n | - | \xi - n | \geq | n | -1$ that

$$\begin{aligned}
&\big\| \widehat{u_n}(\cdot,t) - \hat{u}(\cdot,t) \big\|^2_{L^2(\mathbb{R})} \\
&= \int_{-\infty}^{+\infty} \exp\left(2Re\left(\frac{2\phi_n^v(\xi)\big(T-t\big)}{1+a\phi_n(\xi)} \right) \right) \Big| \widehat{\varphi_n}(\xi) - \widehat{\varphi}(\xi) \Big|^2 \, \mathrm{d}\xi \\
&\geq \int_{|\xi-n|\leq 1} \frac{1}{2n^2} \exp\Big(2c_1 \mid \xi \mid^{\bar{\alpha}} (T-t) \Big) \mathrm{d}\xi \\
&\geq \int_{|\xi-n|\leq 1} \frac{1}{2n^2} \exp\Big(2c_1 (\mid n \mid -1)^{\bar{\alpha}} (T-t) \Big) \mathrm{d}\xi \\
&= \frac{1}{n^2} \exp\Big(2c_1 (\mid n \mid -1)^{\bar{\alpha}} (T-t) \Big).
\end{aligned}$$

Therefore,

$$\big\| \widehat{u_n}(\cdot,t) - \hat{u}(\cdot,t) \big\|_{L^2(\mathbb{R})} \geq \frac{1}{n} \exp\Big(c_1 (\mid n \mid -1)^{\bar{\alpha}} (T-t) \Big).$$

This leads to

$$\lim_{n\to+\infty} \big\| \widehat{u_n}(\cdot,t) - \hat{u}(\cdot,t) \big\|_{L^2(\mathbb{R})} \geq \lim_{n\to+\infty} \frac{1}{n} \exp\Big(c_1 (\mid n \mid -1)^{\bar{\alpha}} (T-t) \Big) = +\infty . \quad (2.3)$$

Combining (2..2) and (2.3), we conclude that problem (1.1) is ill-posed in the sense of Hadamard.

From the condition $(\mathcal{C})$, we can see that the function $\exp\left(\frac{\phi_n^v(\xi)(T-t)}{1+a\phi_n(\xi)} \right)$ have small errors in high frequency components can blow up and completely destroy the solution for $0 \leq t < T$. To overcome the ill-posedness of the problem (1.1), we propose a regularization method, which has been introduced by the authors in the paper [17]. The

cause of ill-posedness, of the problem (1.1) lies in the amplified function $\exp\left(\frac{\phi_n^v(\xi)(T-t)}{1+a\phi_n(\xi)}\right)$ of data for $|\xi| \to +\infty$. To stabilize the problem, a natural idea is to replace this function by a better approximate function, denoted by $\mathbf{F}_\mu(\xi)\exp\left(\frac{\phi_n^v(\xi)(T-t)}{1+a\phi_n(\xi)}\right)$, which should satisfy the following two common properties, as follows

(i) For each fixed $\mu > 0$, $\mathbf{F}_\mu(\xi)\exp\left(\frac{\phi_n^v(\xi)(T-t)}{1+a\phi_n(\xi)}\right)$ is bounded as $|\xi| \to +\infty$.

(ii) If ξ is fixed, the function $\mathbf{F}_\mu(\xi)\exp\left(\frac{\phi_n^v(\xi)(T-t)}{1+a\phi_n(\xi)}\right)$ is close to $\exp\left(\frac{\phi_n^v(\xi)(T-t)}{1+a\phi_n(\xi)}\right)$ as $\mu \to 0$.

Condition 2. Here φ^ϵ is the noisy data of the exact data φ satisfying

$$\left\|\varphi^\epsilon - \varphi\right\|_{H^q\left(\mathbb{R}^n\right)} \le \epsilon, \text{ for some } q \le 0, \qquad (2.7)$$

Since φ^ϵ belongs, in general, to $L^2\left(\mathbb{R}^n\right) \subset H^q\left(\mathbb{R}^n\right)$ for $q \le 0$, so q should not be positive.

We have $\mu := \mu(\epsilon) > 0$ plays the role of regularization parameter. Inspired by the idea of [18], we choose

$$\mathbf{F}_\mu(\xi) := \min\left\{\frac{1}{\mu}\exp\left(-\mathrm{Re}\left(\frac{\phi_n^v(\xi)T}{1+a\phi_n(\xi)}\right)\right), 1\right\} \quad . \qquad (2.4)$$

and introduce the following two regularized solutions for solving the problem (1.1).

The first regularized solution is

$$u_\mu^\epsilon(x,t) = A_1^\epsilon\left(T-t\right)\varphi^\epsilon(\xi) - \int_t^T A_2^\epsilon\left(s-T\right)f(x,s)\mathrm{d}s, \qquad (2.5)$$

with

$$A_1^\epsilon\left(t\right)v\left(x\right) = \frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n}\mathbf{F}_\mu(\xi)\exp\left(\frac{\phi_n^v(\xi)t}{1+a\phi_n(\xi)}\right)\hat{v}(\xi)\mathrm{e}^{ix\cdot\xi}\mathrm{d}\xi,$$

$$A_2^\epsilon\left(t\right)v\left(x\right) = \frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n}\mathbf{F}_\mu(\xi)\frac{1}{1+a\phi_n(\xi)}\exp\left(\frac{\phi_n^v(\xi)t}{1+a\phi_n(\xi)}\right)\hat{v}(\xi)\mathrm{e}^{ix\cdot\xi}\mathrm{d}\xi.$$

The second regularized solutions is

$$w_{\mu}^{\epsilon}(x,t) := A_1^{\epsilon}\left(T-t\right)\varphi^{\epsilon}(\xi) - \int_t^T A_2^{\epsilon}\left(s-T\right)f(x,s)\mathrm{d}s + \int_0^t A_3^{\epsilon}\left(s-t\right)f(x,s)\mathrm{d}s, \quad (2.6)$$

in which

$$A_3^{\epsilon}\left(t\right)v\left(x\right) = \frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n}\left(1-\mathbf{F}_{\mu}(\xi)\right)\frac{1}{1+a\phi_n(\xi)}\exp\left(\frac{\phi_n^{v}(\xi)t}{1+a\phi_n(\xi)}\right)\hat{v}(\xi)\mathrm{e}^{ix\cdot\xi}\mathrm{d}\xi.$$

Under the two regularized solutions (2.5) and (2.6), approximation results are obtained in the following manner.

3. Main results

Definition 3.1. ([17, Definition 3.1]) Assign a solution $\mathcal{U}_{\mu}^{\epsilon}$ as in (2.5) or (2.6). Then the problem (1.1) is regularized by the solution $\mathcal{U}_{\mu}^{\epsilon}$ if for any $\epsilon > 0$, there exists $\mu = \mu(\epsilon) > 0$ such that

(i) $\mu \to 0\, as\, \epsilon \to 0^+$,

$(ii)\left\|\mathcal{U}_{\mu}^{\epsilon}(\cdot,t) - u(\cdot,t)\right\|_{H^q\left(\mathbb{R}^n\right)} \to 0$ as $\epsilon \to 0^+$ for $t \in [0,T)$ whenever $\left\|\varphi^{\epsilon} - \varphi\right\|_{H^q\left(\mathbb{R}^n\right)} \le \epsilon$.

Theorem 3.1 Assume a nonhomogenous $f(\cdot,t) \in H^q\left(\mathbb{R}^n\right)$ for all $t \in [0,T]$ and suppose that the problem (1.1) has a unique solution $u \in C\left([0,T], H^q\left(\mathbb{R}^n\right)\right)$.

$i)$ Let u satisfy an *a priori* bound condition

$$\left(\int_{\mathbb{R}^n}\left(1+|\,\xi\,|^2\right)^p\left|\exp\left(\frac{\phi_n^{v}(\xi)t}{1+a\phi_n(\xi)}\right)\hat{u}(\xi,t)\right|^2\mathrm{d}\xi\right)^{\frac{1}{2}} \le E \text{ for some } p \ge q. \quad (3.1)$$

If the regularization parameter is chosen by

$$\mu = \frac{\epsilon}{E}\left(\frac{1}{T}\ln\frac{E}{\epsilon}\right)^{\frac{p-q}{\bar{\alpha}}},$$

then for every $t \in [0,T)$, the H^q - norm of the error between the regularized solution u_{μ}^{ϵ} given by (2.5) and the exact solution u is estimated by

$$\left\|u_{\mu}^{\epsilon}(\cdot,t) - u(\cdot,t)\right\|_{H^q\left(\mathbb{R}^n\right)} \le CE^{1-\frac{t}{T}}\epsilon^{\frac{t}{T}} \times \left(\frac{1}{T}\ln\frac{E}{\epsilon}\right)^{-\frac{(p-q)(T-t)}{\bar{\alpha}T}}(2+o(1)),\ \epsilon \to 0^+. \quad (3.2)$$

ii) Let u satisfy an *a priori* bound condition

$$\| u(\cdot,0) \|_{H^p\left(\mathbb{R}^n\right)} \leq E_0 \text{ for some } p \geq q. \qquad (3.3)$$

If $KT < 1$ and the regularization parameter is selected by

$$\mu = \frac{\epsilon}{E_0}\left(\frac{1}{T}\ln\frac{E_0}{\epsilon}\right)^{\frac{p-q}{\bar{\alpha}}},$$

then for every $t \in [0,T)$, the H^q-error estimate between the regularized solution w_μ^ϵ given by (2.6) and the exact solution u is given by

$$\left\|w_\mu^\epsilon(\cdot,t) - u(\cdot,t)\right\|_{H^q\left(\mathbb{R}^n\right)} \leq \sqrt{\left(1+1/a\right)C}\, E_0^{1-\frac{t}{T}}\epsilon^{\frac{t}{T}}\left(\frac{1}{T}\ln\frac{E_0}{\epsilon}\right)^{-\frac{(p-q)(T-t)}{\bar{\alpha}T}}(2+o(1)), \epsilon \to 0^+. \quad (3.4)$$

4. The proof of Theorem 3.1

4.1. Some auxiliary results

To prove **Theorem 3.1** we need the following two lemmas.

Lemma 4.1. Let $\mathbf{F}_\mu$ be as in and $\mu \in (0,1)$. Then, the following estimates hold.

$$i)\sup_{\xi\in\mathbb{R}^n}\left|\mathbf{F}_\mu(\xi)\exp\left(\frac{\phi_n^v(\xi)(T-t)}{1+a\phi_n(\xi)}\right)\right| \leq \mu^{-\frac{(T-t)}{T}},\ for\ all\ 0 \leq t \leq T.$$

$$ii)\sup_{\xi\in\mathbb{R}^n}\left|\mathbf{F}_\mu(\xi)\exp\left(\frac{\phi_n^v(\xi)(s-t)}{1+a\phi_n(\xi)}\right)\right| \leq \mu^{-\frac{(s-t)}{T}},\ for\ all\ 0 \leq t \leq s \leq T.$$

$$iii)\sup_{\xi\in\mathbb{R}^n}\left|\left(1-\mathbf{F}_\mu(\xi)\right)\exp\left(-\frac{\phi_n^v(\xi)(t-s)}{1+a\phi_n(\xi)}\right)\right| \leq \mu^{\frac{(t-s)}{T}},\ for\ all\ 0 \leq s \leq t \leq T.$$

iv) For $r \geq 0$, *then*

$$\sup_{\xi\in\mathbb{R}^n}\langle\xi\rangle^{-r}\left|\left(1-\mathbf{F}_\mu(\xi)\right)\exp\left(-\frac{\phi_n^v(\xi)t}{1+a\phi_n(\xi)}\right)\right| \leq c_2^{\frac{r}{\bar{\alpha}}}\left(\frac{1}{T}\ln\frac{1}{\mu}\right)^{-\frac{r}{\bar{\alpha}}}\mu^{\frac{t}{T}},\ for\ all\ 0 \leq t \leq T,$$

with $\bar{\alpha}$, c_2 as in the condition $(\mathcal{C})$.

Proof.

i) A straightforward computation using (2.4) and the fact $\left|e^z\right| = e^{\mathrm{Re}(z)}, z \in \mathbb{C}$ shows that

$$\sup_D\left|\mathbf{F}_\mu(\xi)\exp\left(\frac{\phi_n^v(\xi)(T-t)}{1+a\phi_n(\xi)}\right)\right| = \sup_D\frac{1}{\mu}\exp\left(-\mathrm{Re}\left(\frac{\phi_n^v(\xi)t}{1+a\phi_n(\xi)}\right)\right) \leq \mu^{-\frac{(T-t)}{T}},$$

where $D = \left\{ \xi \in \mathbb{R}^n : \frac{1}{\mu} < \exp\left(\mathrm{Re}\left(\frac{\phi_n^v(\xi) T}{1 + a\phi_n(\xi)} \right) \right) \right\}$.

On the other hand, one has

$$\sup_{\bar{D}} \left| \mathbf{F}_\mu(\xi) \exp\left(\frac{\phi_n^v(\xi)(T-t)}{1 + a\phi_n(\xi)} \right) \right| = \sup_{\bar{D}} \left(\exp\left(\mathrm{Re}\left(\frac{\phi_n^v(\xi) T}{1 + a\phi_n(\xi)} \right) \right) \right)^{\frac{T-t}{T}} \leq \mu^{-\frac{(T-t)}{T}},$$

where $\bar{D} = \left\{ \xi \in \mathbb{R}^n : \frac{1}{\mu} \geq \exp\left(\mathrm{Re}\left(\frac{\phi_n^v(\xi) T}{1 + a\phi_n(\xi)} \right) \right) \right\}$.

Hence, $i)$ holds.

ii) The proof of $ii)$ is similar to $i)$. Thus, we omit it here.

iii) As the proof in $i)$, from (2.4), we get

$$\begin{aligned}
&\sup_{\xi \in \mathbb{R}^n} \left| \left(1 - \mathbf{F}_\mu(\xi)\right) \exp\left(-\frac{\phi_n^v(\xi)(t-s)}{1 + a\phi_n(\xi)} \right) \right| \\
&\leq \sup_D \left| \left(1 - \frac{1}{\mu} \exp\left(-\mathrm{Re}\left(\frac{\phi_n^v(\xi) T}{1 + a\phi_n(\xi)} \right) \right) \right) \right| \exp\left(-\mathrm{Re}\left(\frac{\phi_n^v(\xi)(t-s)}{1 + a\phi_n(\xi)} \right) \right) \\
&\leq \sup_D \exp\left(-\mathrm{Re}\left(\frac{\phi_n^v(\xi)(t-s)}{1 + a\phi_n(\xi)} \right) \right) \leq \mu^{\frac{t-s}{T}}.
\end{aligned}$$

Thus, $iii)$ holds.

iv) Since $\mathbf{F}_\mu$ is given by $\left(2.4\right)$ and $\langle \xi \rangle^{-r} \leq |\, \xi \,|^{-r}, \forall r \geq 0$, we have

$$\begin{aligned}
&\sup_{\xi \in \mathbb{R}^n} \langle \xi \rangle^{-r} \left| \left(1 - \mathbf{F}_\mu(\xi)\right) \exp\left(-\frac{\phi_n^v(\xi) t}{1 + a\phi_n(\xi)} \right) \right| \\
&\leq \sup_D \left| \xi \right|^{-r} \left| 1 - \frac{1}{\mu} \exp\left(-Re\left(\frac{\phi_n^v(\xi) T}{1 + a\phi_n(\xi)} \right) \right) \right| \exp\left(-Re\left(\frac{\phi_n^v(\xi) t}{1 + a\phi_n(\xi)} \right) \right) \\
&\leq \sup_D \left| \xi \right|^{-r} \exp\left(-\mathrm{Re}\left(\frac{\phi_n^v(\xi) t}{1 + a\phi_n(\xi)} \right) \right).
\end{aligned}$$

Now the condition $(\mathcal{C})$ leads to $|\, \xi \,|^{-r} \leq \left(\frac{1}{c_2} \mathrm{Re}\left(\frac{\phi_n^v(\xi)}{1 + a\phi_n(\xi)} \right) \right)^{-\frac{r}{\bar{\alpha}}}$.

Consequently,

$$\sup_{\xi\in\mathbb{R}^n}\langle\xi\rangle^{-r}\left|\Big(1-\mathbf{F}_{\mu}(\xi)\Big)\exp\left(-\frac{\phi_n^v(\xi)t}{1+a\phi_n(\xi)}\right)\right|$$
$$\leq c_2^{\frac{r}{\bar{\alpha}}}\sup_{D}\left|\mathrm{Re}\left(\frac{\phi_n^v(\xi)}{1+a\phi_n(\xi)}\right)\right|^{-\frac{r}{\bar{\alpha}}}\exp\left(-\mathrm{Re}\left(\frac{\phi_n^v(\xi)t}{1+a\phi_n(\xi)}\right)\right)$$
$$\leq c_2^{\frac{r}{\bar{\alpha}}}\left(\frac{1}{T}\ln\frac{1}{\mu}\right)^{-\frac{r}{\bar{\alpha}}}\mu^{\frac{t}{T}}.$$

Hence, $iv)$ holds.

The proof of Lemma 4.1 is complete.

Lemma 4.2. Let the function $f(\beta):(0,\delta]\to\mathbb{R}$ be given by $f(\beta)=\beta^b\left(d\ln\frac{1}{\beta}\right)^{-m}$, with a constant $m\in\mathbb{R}$ and positive constants $\delta<1$, b and d, then for the inverse function we have

$$f^{-1}(\beta)=\beta^{1/b}\left(\frac{d}{b}\ln\frac{1}{\beta}\right)^{m/b}(1+o(1)),\ \text{for}\ \beta\to 0^+.$$

Proof. See Lemma 3.3 in [19].

4.2. The proof of Theorem 3.1

We first give two following propositions which are useful to the proof of Theorem 3.1.

Proposition 4.1. Let f be as in Theorem 3.1, $u_{\mu,1}^{\epsilon},u_{\mu,2}^{\epsilon}$ be the solutions of (2.7) corresponding to the data $\varphi_1^{\epsilon},\varphi_2^{\epsilon}$, respectively, then for all $t\in[0,T]$, we have the following stability estimate

$$\left\|u_{\mu,1}^{\epsilon}(\cdot,t)-u_{\mu,2}^{\epsilon}(\cdot,t)\right\|_{H^q\left(\mathbb{R}^n\right)}\leq\mu^{-\frac{(T-t)}{T}}\left\|\varphi_1^{\epsilon}-\varphi_2^{\epsilon}\right\|_{H^q\left(\mathbb{R}^n\right)}.\ \Big(4.1\Big)$$

Proof.

The stability of solution of the integral equation (4.1).

Using (2.5),

$$\hat{u}(\xi,t)=\exp\left(\frac{\phi_n^v(\xi)\Big(T-t\Big)}{1+a\phi_n(\xi)}\right)\hat{\varphi}(\xi)-\int_t^T\exp\left(\frac{\phi_n^v(\xi)\Big(s-t\Big)}{1+a\phi_n(\xi)}\right)\hat{f}(\xi,s)\mathrm{d}s,$$

we find that

$$\widehat{u^{\epsilon}_{\mu,1}}(\xi,t)-\widehat{u^{\epsilon}_{\mu,2}}(\xi,t)=\mathbf{F}_{\mu}(\xi)\exp\left(\frac{\phi_n^{v}(\xi)(T-t)}{1+a\phi_n(\xi)}\right)\left(\widehat{\varphi_1^{\epsilon}}(\xi)-\widehat{\varphi_2^{\epsilon}}(\xi)\right).$$

Then, we have

$$\left\|u^{\epsilon}_{\mu,1}(\cdot,t)-u^{\epsilon}_{\mu,2}(\cdot,t)\right\|^2_{H^q\left(\mathbb{R}^n\right)}\leq\int_{\mathbb{R}^n}\langle\xi\rangle^{2q}\left|\mathbf{F}_{\mu}(\xi)\exp\left(\frac{\phi_n^{v}(\xi)(T-t)}{1+a\phi_n(\xi)}\right)\left(\widehat{\varphi_1^{\epsilon}}(\xi)-\widehat{\varphi_2^{\epsilon}}(\xi)\right)\right|^2\mathrm{d}\xi.\ \left(4.2\right)$$

For the first term on the right side of (4.2), by Lemma 4.1 i), we get

$$\int_{\mathbb{R}^n}\langle\xi\rangle^{2q}\left|\mathbf{F}_{\mu}(\xi)\exp\left(\frac{\phi_n^{v}(\xi)(T-t)}{1+a\phi_n(\xi)}\right)\left(\widehat{\varphi_1^{\epsilon}}(\xi)-\widehat{\varphi_2^{\epsilon}}(\xi)\right)\right|^2\mathrm{d}\xi\leq\mu^{-\frac{2(T-t)}{T}}\left\|\varphi_1^{\epsilon}-\varphi_2^{\epsilon}\right\|^2_{H^q\left(\mathbb{R}^n\right)}.$$

In summary, from $\left(4.2\right)$ we get

$$\left\|u^{\epsilon}_{\mu,1}(\cdot,t)-u^{\epsilon}_{\mu,2}(\cdot,t)\right\|^2_{H^q\left(\mathbb{R}^n\right)}\leq\mu^{-\frac{2(T-t)}{T}}\left\|\varphi_1^{\epsilon}-\varphi_2^{\epsilon}\right\|^2_{H^q\left(\mathbb{R}^n\right)}.$$

Therefore,

$$\left\|u^{\epsilon}_{\mu,1}(\cdot,t)-u^{\epsilon}_{\mu,2}(\cdot,t)\right\|_{H^q\left(\mathbb{R}^n\right)}\leq\mu^{-\frac{(T-t)}{T}}\left\|\varphi_1^{\epsilon}-\varphi_2^{\epsilon}\right\|_{H^q\left(\mathbb{R}^n\right)}.$$

This completes the proof of Proposition 4.1.

Proposition 4.2. Let f be as in Theorem 3.1, $w^{\epsilon}_{\mu,1},w^{\epsilon}_{\mu,2}$ be the solutions of (2.6) corresponding to the data $\varphi_1^{\epsilon},\varphi_2^{\epsilon}$, respectively, then we get the following stability estimate

$$\left\|w^{\epsilon}_{\mu,1}(\cdot,t)-w^{\epsilon}_{\mu,2}(\cdot,t)\right\|_{H^q\left(\mathbb{R}^n\right)}\leq\sqrt{(1+1/a)}\mu^{-\frac{(T-t)}{T}}\left\|\varphi_1^{\epsilon}-\varphi_2^{\epsilon}\right\|_{H^q\left(\mathbb{R}^n\right)},\forall t\in[0,T],$$

where $0<a<\dfrac{1}{K^2T^2}-1$.

Proof.

The proof of Proposition 4.2 is shown similarly to the proof of the above one.

Finally, we show Proof of Theorem 3.1.

(1) Let us define u_{μ} be the regularized solution as in (2.5) with respect to the exact data φ. By the triangle inequality yields that

$$\left\|u_\mu^\epsilon(\cdot,t)-u(\cdot,t)\right\|_{H^q\left(\mathbb{R}^n\right)} \le \left\|u_\mu^\epsilon(\cdot,t)-u_\mu(\cdot,t)\right\|_{H^q\left(\mathbb{R}^n\right)} + \left\|u_\mu(\cdot,t)-u(\cdot,t)\right\|_{H^q\left(\mathbb{R}^n\right)} := I_1 + I_2. \quad (4.3)$$

We now estimate the terms $I_i,\ i=1,2$

Estimation of I_1 : Using Proposition 4.1 and (2.5), we find that

$$I_1 = \left\|u_\mu^\epsilon(\cdot,t)-u_\mu(\cdot,t)\right\|_{H^q\left(\mathbb{R}^n\right)} \le \mu^{-\frac{(T-t)}{T}}\epsilon\ . \quad (4.4)$$

Estimation of I_2 : A straightforward computation using (2.1) and (2.5) shows that

$$\widehat{u}(\xi,t)-\widehat{u_\mu}(\xi,t) = \left(1-\mathbf{F}_\mu(\xi)\right)\left(\exp\left(\frac{\phi_n^v(\xi)(T-t)}{1+a\phi_n(\xi)}\right)\hat{\varphi}(\xi) - \int_t^T \exp\left(\frac{\phi_n^v(\xi)(s-t)}{1+a\phi_n(\xi)}\right)\hat{f}(\xi,s)\mathrm{d}s\right).$$

and

$$\left\| u(\cdot,t)-u_\mu(\cdot,t)\right\|^2_{H^q\left(\mathbb{R}^n\right)}$$

$$= \int_{\mathbb{R}^n}\langle\xi\rangle^{2q}\left|1-\mathbf{F}_\mu(\xi)\right|^2\left(\exp\left(\frac{\phi_n^v(\xi)(T-t)}{1+a\phi_n(\xi)}\right)\hat{\varphi}(\xi) - \int_t^T \exp\left(\frac{\phi_n^v(\xi)(s-t)}{1+a\phi_n(\xi)}\right)\hat{f}(\xi,s)\mathrm{d}s\right)^2 \mathrm{d}\xi. \quad (4.5)$$

Next using (2.1), (2.5) and Lemma 4.1 iv) we have

$$\int_{\mathbb{R}^n}\langle\xi\rangle^{2q}\left|1-\mathbf{F}_\mu(\xi)\right|^2\left(\exp\left(\frac{\phi_n^v(\xi)(T-t)}{1+a\phi_n(\xi)}\right)\hat{\varphi}(\xi) - \int_t^T \exp\left(\frac{\phi_n^v(\xi)(s-t)}{1+a\phi_n(\xi)}\right)\hat{f}(\xi,s)\mathrm{d}s\right)^2 \mathrm{d}\xi$$

$$= \int_{\mathbb{R}^n}\langle\xi\rangle^{-2(p-q)}\left|\left(1-\mathbf{F}_\mu(\xi)\right)\exp\left(-\frac{\phi_n^v(\xi)t}{1+a\phi_n(\xi)}\right)\right|^2\langle\xi\rangle^{2p}\left|\exp\left(\frac{\phi_n^v(\xi)t}{1+a\phi_n(\xi)}\right)\hat{u}(\xi,t)\right|^2 \mathrm{d}\xi$$

$$\le CE^2\left(\frac{1}{T}\ln\frac{1}{\mu}\right)^{-\frac{2(p-q)}{\bar{\alpha}}}\mu^{\frac{2t}{T}}. \quad (4.6)$$

Combining with (4.5) and (4.6), we find that

$$\left\|u(\cdot,t)-u_\mu(\cdot,t)\right\|^2_{H^q\left(\mathbb{R}^n\right)} \le\ CE^2\left(\frac{1}{T}\ln\frac{1}{\mu}\right)^{-\frac{2(p-q)}{\bar{\alpha}}}\mu^{\frac{2t}{T}} \quad (4.7)$$

Inserting (4.4) and (4.7) in (4.3), we infer

$$\left\|u_\mu^\epsilon(\cdot,t)-u(\cdot,t)\right\|_{H^q\left(\mathbb{R}^n\right)} \le C\left(\mu^{\frac{-(T-t)}{T}} + E\mu^{\frac{t}{T}}\left(\frac{1}{T}\ln\frac{1}{\mu}\right)^{-\frac{p-q}{\bar{\alpha}}}\right) \quad (4.8)$$

Now we can choose the regularization parameter μ according to

$$\mu^{-\frac{(T-t)}{T}}\epsilon = E\mu^{\frac{t}{T}}\left(\frac{1}{T}\ln\frac{1}{\mu}\right)^{-\frac{p-q}{\overline{\alpha}}}, i.e., \mu\left(\frac{1}{T}\ln\frac{1}{\mu}\right)^{-\frac{p-q}{\overline{\alpha}}} = \frac{\epsilon}{E}. \quad (4.9)$$

Applying Lemma 4.2 with $b = 1,\ d = \frac{1}{T}$ and $m = \frac{p-q}{\overline{\alpha}}$, we obtain

$$\mu = \frac{\epsilon}{E}\left(\frac{1}{T}\ln\frac{E}{\epsilon}\right)^{\frac{p-q}{\overline{\alpha}}}(1+o(1)),\ \epsilon \to 0^+.$$

Hence

$$\mu^{-\frac{(T-t)}{T}}\epsilon = E^{1-\frac{t}{T}}\epsilon^{\frac{t}{T}}\left(\frac{1}{T}\ln\frac{E}{\epsilon}\right)^{-\frac{(p-q)(T-t)}{\overline{\alpha}T}}$$

and

$$E\mu^{\frac{t}{T}}\left(\frac{1}{T}\ln\frac{1}{\mu}\right)^{-\frac{p-q}{\overline{\alpha}}} = E\left(\frac{E}{\epsilon}\left(\frac{1}{T}\ln\frac{E}{\epsilon}\right)^{-\frac{p-q}{\overline{\alpha}}}\right)^{-\frac{t}{T}}\left(\frac{1}{T}\ln\left(\frac{E}{\epsilon}\left(\frac{1}{T}\ln\frac{E}{\epsilon}\right)^{-\frac{p-q}{\overline{\alpha}}}\right)\right)^{-\frac{p-q}{\overline{\alpha}}}$$

$$= E\left(\frac{\epsilon}{E}\right)^{\frac{t}{T}}\left(\frac{1}{T}\ln\frac{E}{\epsilon}\right)^{\frac{(p-q)t}{\overline{\alpha}T}}\left(\frac{1}{T}\ln\frac{E}{\epsilon}+\frac{1}{T}\ln\left(\frac{1}{T}\ln\frac{E}{\epsilon}\right)^{-\frac{p-q}{\overline{\alpha}}}\right)^{-\frac{p-q}{\overline{\alpha}}}$$

$$= E^{1-\frac{t}{T}}\epsilon^{\frac{t}{T}}\left(\frac{1}{T}\ln\frac{E}{\epsilon}\right)^{-\frac{(p-q)(T-t)}{\overline{\alpha}T}}\left(1+\frac{\ln\left(\frac{1}{T}\ln\frac{E}{\epsilon}\right)^{-\frac{p-q}{\overline{\alpha}}}}{\ln\frac{E}{\epsilon}}\right)^{-\frac{p-q}{\overline{\alpha}}}$$

$$= E^{1-\frac{t}{T}}\epsilon^{\frac{t}{T}}\left(\frac{1}{T}\ln\frac{E}{\epsilon}\right)^{-\frac{(p-q)(T-t)}{\overline{\alpha}T}}(1+o(1)), \epsilon \to 0^+. \quad (4.10)$$

Inserting (4.9) and (4.10) into (4.8) yields

$$\left\|u_\mu^\epsilon(\cdot,t)-u(\cdot,t)\right\|_{H^q\left(\mathbb{R}^n\right)} \le CE^{1-\frac{t}{T}}\epsilon^{\frac{t}{T}}\left(\frac{1}{T}\ln\frac{E}{\epsilon}\right)^{-\frac{(p-q)(T-t)}{\overline{\alpha}T}}(2+o(1)), \epsilon \to 0^+ \quad (4.11)$$

This completes the proof of Theorem 3.1 i).

Next, by using (2.1), it is straightforward to check that

$$\hat{\varphi}(\xi)=\exp\left(-\frac{\phi_n^v(\xi)T}{1+a\phi_n(\xi)}\right)\left(\hat{u}(\xi,0)+\int_0^T\exp\left(\frac{\phi_n^v(\xi)s}{1+a\phi_n(\xi)}\right)\hat{f}(\xi,s)\mathrm{d}s\right)$$

which gives

$$\begin{aligned}&\hat{\varphi}(\xi)-\int_t^T\exp\left(\frac{\phi_n^v(\xi)(s-T)}{1+a\phi_n(\xi)}\right)\hat{f}(\xi,s)\mathrm{d}s\\&=\exp\left(-\frac{\phi_n^v(\xi)T}{1+a\phi_n(\xi)}\right)\widehat{u(\cdot,0)}(\xi)+\int_0^t\exp\left(-\frac{\phi_n^v(\xi)(s-T)}{1+a\phi_n(\xi)}\right)\hat{f}(\xi,s)\mathrm{d}s.\end{aligned}$$

Combining with (2.1), we obtain

$$\begin{aligned}&\hat{u}(\xi,t)\\&=\left(\exp\left(\frac{\phi_n^v(\xi)(T-t)}{1+a\phi_n(\xi)}\right)\hat{\varphi}(\xi)-\int_t^T\exp\left(\frac{\phi_n^v(\xi)(s-t)}{1+a\phi_n(\xi)}\right)\hat{f}(\xi,s)\mathrm{d}s\right)\mathbf{F}_\mu(\xi)\\&+\exp\left(\frac{\phi_n^v(\xi)(T-t)}{1+a\phi_n(\xi)}\right)\left(\hat{\varphi}(\xi)-\int_t^T\exp\left(\frac{\phi_n^v(\xi)(s-T)}{1+a\phi_n(\xi)}\right)\hat{f}(\xi,s)\mathrm{d}s\right)\left(1-\mathbf{F}_\mu(\xi)\right)\\&=\left(\exp\left(\frac{\phi_n^v(\xi)(T-t)}{1+a\phi_n(\xi)}\right)\hat{\varphi}(\xi)-\int_t^T\exp\left(\frac{\phi_n^v(\xi)(s-t)}{1+a\phi_n(\xi)}\right)\hat{f}(\xi,s)\mathrm{d}s\right)\mathbf{F}_\mu(\xi)\\&+\left(1-\mathbf{F}_\mu(\xi)\right)\exp\left(\frac{-\phi_n^v(\xi)t}{1+a\phi_n(\xi)}\right)\hat{u}(\xi,0)+\left(1-\mathbf{F}_\mu(\xi)\right)\int_0^t\exp\left(\frac{-\phi_n^v(\xi)(s-t)}{1+a\phi_n(\xi)}\right)\hat{f}(\xi,s)\mathrm{d}s.\end{aligned}$$

Let w_μ be the regularized solution as in (2.6) with respect to the exact data φ. By using (2.6) and the triangle inequality, we obtain for all $t\in[0,T]$ that

$$\left|\widehat{w_\mu}(\xi,t)-\hat{u}(\xi,t)\right|\le\left|\left(1-\mathbf{F}_\mu(\xi)\right)\exp\left(-\frac{\phi_n^v(\xi)t}{1+a\phi_n(\xi)}\right)\right|\,|\,\hat{u}(\xi,0)\,|\,. \qquad (4.13)$$

Thus, the inequality **Lemma 4.1 iv)** and (2.7) imply that

$$\begin{aligned}&\left\|w_\mu(\cdot,t)-u(\cdot,t)\right\|^2_{H^q(\mathbb{R}^n)}\\&\le(1+1/a)\sup_{\xi\in\mathbb{R}^n}\langle\xi\rangle^{-2(p-q)}\left|\left(1-\mathbf{F}_\mu(\xi)\right)\exp\left(-\frac{\phi_n^v(\xi)t}{1+a\phi_n(\xi)}\right)\right|^2\ \|\,u(\cdot,0)\,\|^2_{H^p(\mathbb{R}^n)}\\&\le(1+1/a)C\left(\frac{1}{T}\ln\frac{1}{\mu}\right)^{-\frac{2(p-q)}{\alpha}}\ \mu^{\frac{2t}{T}}E_0^2.\end{aligned}$$

This leads to

$$\left\|w_{\mu}(\cdot,t)-u(\cdot,t)\right\|_{H^q\left(\mathbb{R}^n\right)} \leq \sqrt{(1+1/a)C}\,\mu^{\frac{t}{T}}\left(\frac{1}{T}\ln\frac{1}{\mu}\right)^{-\frac{(p-q)}{\bar{\alpha}}} E_0 \qquad (4.14)$$

Thanks to the triangle inequality, Proposition 4.2, (2.7) and (4.14), we get that

$$\begin{aligned}
&\left\|w_{\mu}^{\epsilon}(\cdot,t)-u(\cdot,t)\right\|_{H^q\left(\mathbb{R}^n\right)} \\
&\leq \left\|w_{\mu}^{\epsilon}(\cdot,t)-w_{\mu}(\cdot,t)\right\|_{H^q\left(\mathbb{R}^n\right)} + \left\|w_{\mu}(\cdot,t)-u(\cdot,t)\right\|_{H^q\left(\mathbb{R}^n\right)} \\
&\leq \sqrt{(1+1/a)C}\left(\mu^{-\frac{(T-t)}{T}}\epsilon + \mu^{\frac{t}{T}}\left(\frac{1}{T}\ln\frac{1}{\mu}\right)^{-\frac{p-q}{\bar{\alpha}}} E_0\right).
\end{aligned}$$

Then, proceeding similarly to (4.11), we obtain

$$\left\|w_{\mu}^{\epsilon}(\cdot,t)-u(\cdot,t)\right\|_{H^q\left(\mathbb{R}^n\right)} \leq \sqrt{\left(1+1/a\right)C}\; E_0^{1-\frac{t}{T}}\epsilon^{\frac{t}{T}}\left(\frac{1}{T}\ln\frac{E_0}{\epsilon}\right)^{-\frac{(p-q)(T-t)}{\bar{\alpha}T}} (2+o(1)), \epsilon \to 0^+$$

This completes the proof of Theorem 3.1 ii).

5. Conclusions

In this paper, two modified versions of the so-called optimal filtering method of Seidman are proposed to deal with the backward problem for a fractional pseudo - parabolic equation with a nonhomogenous source term. Under different *a priori* assumptions on the regularity of the exact solution, a fractional pseudo - parabolic equation with a nonhomogenous source term rate of convergence in the Sobolev space $H^q\left(\mathbb{R}^n\right)$ has been obtained.

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