

A BACKWARD HEAT EQUATION IN SPHERICAL COORDINATES WITH NONLINEAR HEAT SOURCE

Bài toán nhiệt ngược trong tọa độ cầu với nguồn nhiệt phi tuyến

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Abstract

The problem of determining the initial temperature distribution in the backward heat equation (BHP) has attracted significant attention across various physical disciplines. While numerous studies have addressed the linear version of this inverse problem, the nonlinear case remains largely unexplored. To the best of our knowledge, the literature contains only limited results regarding the nonlinear BHP formulated in spherical coordinates. Motivated by this research gap, the present work aims to analyze the nonlinear backward heat equation in spherical coordinates. We begin by examining the inherent ill-posedness of the problem and then propose a regularization strategy based on the quasi-boundary value method to obtain a stable approximate solution. Our analysis demonstrates that the regularized solution not only exhibits continuous dependence on the terminal data but also converges to the exact solution in an appropriate function space.

Keywords: *Nonlinear backward heat equation; Inverse problem; Spherical coordinates; Regularization method; Quasi-boundary value method.*

Tóm tắt

Bài toán xác định phân bố nhiệt độ ban đầu trong phương trình nhiệt ngược (BHE) đã thu hút nhiều sự quan tâm trong các lĩnh vực vật lý khác nhau. Mặc dù đã có nhiều nghiên cứu tập trung vào trường hợp tuyến tính của bài toán ngược này, trường hợp phi tuyến vẫn còn ít được khai thác. Đặc biệt, theo hiểu biết của chúng tôi, hiện có rất ít kết quả về bài toán BHE phi tuyến được xây dựng trong hệ tọa độ cầu. Xuất phát từ điều này, kết quả nghiên cứu hiện tại hướng đến việc phân tích phương trình nhiệt ngược phi tuyến trong tọa độ cầu. Chúng tôi bắt đầu bằng việc khảo sát tính không chỉnh của bài toán, sau đó đề xuất một phương pháp chỉnh hóa dựa trên phương pháp tựa giá trị biên để thu được nghiệm gần đúng ổn định. Chúng tôi cũng chứng minh được nghiệm chỉnh hóa không chỉ phụ thuộc liên tục vào dữ liệu cuối mà còn hội tụ về nghiệm chính xác trong không gian thích hợp.

Từ khóa: *Bài toán nhiệt ngược phi tuyến; Bài toán ngược; Tọa độ cầu; Phương pháp chỉnh hóa; Phương pháp tựa giá trị biên.*

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1. Introduction

Let a , T , and c be positive real numbers, and let $\Omega = (0, a) \times (0, \pi) \times (0, 2\pi)$. We consider the inverse problem for the following nonlinear parabolic equation

$$u_t(r, \theta, \phi, t) = c^2 \nabla^2 u(r, \theta, \phi, t) + q(r, \theta, \phi, t, u(r, \theta, \phi, t)), \quad (1.1)$$

$$u(a, \theta, \phi, t) = 0, \quad (1.2)$$

$$u(r, \theta, \phi, T) = f(r, \theta, \phi), \quad (1.3)$$

$$|u(0, \theta, \phi, t)| < \infty, \quad (1.4)$$

where ∇^2 denotes the Laplace operator in spherical coordinates, given by

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \left(\frac{\partial^2 u}{\partial \theta^2} + \cot \theta \frac{\partial u}{\partial \theta} + \csc^2 \theta \frac{\partial^2 u}{\partial \phi^2} \right).$$

The constant a is the radius of the ball, while r is the radial coordinate. Throughout this paper, we use the Hilbert space $H = L^2(\Omega; r^2 \sin \theta dr d\theta d\phi)$. The final temperature f is assumed to belong to H . For brevity, we write

$$q(u)(r, \theta, \phi, t) = q(r, \theta, \phi, t, u(r, \theta, \phi, t)).$$

The precise assumptions on the nonlinear heat source q will be stated in Section 3.

As we know, the aim of the above problem is to determine the unknown initial temperature distribution at any earlier time $t < T$ from the temperature profile at a particular time $t = T$ (the final data). This problem is called the backward heat problem (BHP), or the final-value problem. In practical applications, many physical processes can be described by the BHP, including heat propagation in thermophysics and continuum mechanics. In general, this problem is severely ill-posed in Hadamard's sense. This means that a solution does not always exist, and even when it does, it does not depend continuously on the given data. In particular, the final data f are obtained through physical measurements, which inevitably involve errors. Therefore, in practice, we only have the measured data f^ε such that

$$\|f^\varepsilon - f\|_H \leq \varepsilon,$$

where $\varepsilon > 0$ is the noise level.

This implies that the solution corresponding to the measured data f^ε has large errors from noise level ε . Thus, an appropriate regularization method needs to be proposed in order to get a stable solution. Until now, many methods, presented in several articles (see in [3, 6, 7, 13, 14] and the references therein), have been proposed to deal with the BHP. However, the above results are mainly devoted to the linear case of the BHP. Recently, the nonlinear case of the BHP, described by the following problem, has also been considered by some authors (see in [8, 11, 12, 15])

$$\begin{aligned} a(x, t)u_t &= u_{xx} + u_{yy} + g(x, y, t, u(x, y, t)), & (x, y, t) \in I \times (0, T), \\ u(0, y, t) &= u(\pi, y, t) = u(x, 0, t) = u(x, \pi, t) = 0, & t \in (0, T), \\ u(x, y, T) &= \varphi(x, y), & (x, y) \in I, \end{aligned} \quad (1.5)$$

where $I = (0, \pi) \times (0, \pi)$ or $\mathbb{R}^2$, $a(x, t)$ is a thermal conductivity and the functions φ , g are given. In [13, 16, 17, 20], various regularization methods were applied to obtain the stable approximate solutions such as the truncation method, the quasi-reversibility method, the quasi-boundary value method and the eigenfunction expansion method.

Recently, several studies have considered different domains for the BHP, such as in polar or cylindrical coordinates. In particular, Cheng W. et al. studied an axisymmetric homogeneous backward heat problem inside a ball of radius r_0 , in [1]. In that work, the authors investigated the problem of determining the initial temperature distribution in a spherically symmetric parabolic equation as follows:

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial r^2} - \frac{2}{r} \frac{\partial u}{\partial r} = 0, & 0 < r \leq r_0, \quad 0 < t < T, \\ u(r, T) = g(r), & 0 \leq r \leq r_0, \\ u_r(r_0, t) = 0, & 0 \leq t \leq T, \\ |u(0, t)| < \infty, & 0 \leq t < T. \end{cases} \quad (1.6)$$

Using Tikhonov regularization method, the authors dealt with the problem (1.6) and got the logarithmic type of error estimate. It is remarkable that $g(r)$ in problem (1.6) is radially symmetric or spherically symmetric, i.e., it depends only on the radius r and not on θ, ϕ . Therefore, the solution of problem (1.6) does not depend on θ, ϕ , but in practical applications, these things always do not happen. Very recently, Triet L. M. et al. [10] considered the asymmetric backward heat problem in a ball which is more general than the problem (1.6). In particular, the authors dealt with the following parabolic equation

$$\begin{aligned} u_t &= c^2 \nabla^2 u + q(r, \theta, \phi), \\ u(a, \theta, \phi, t) &= 0, \\ u(r, \theta, \phi, T) &= f(r, \theta, \phi), \\ |u(0, \theta, \phi, t)| &< \infty, \end{aligned} \quad (1.7)$$

in which ∇^2 denotes the spherical form of Laplacian. Moreover, the Hölder type of the error estimate was given by applying the modified quasi-boundary value (MQBV) method.

In the above problems (1.6)-(1.7), we show that the BHP is only considered in the symmetric case or the linear case. In fact, by practical measurements we cannot always

obtain symmetric or axisymmetric data of f . Additionally, the heat source q is not often linear. In our knowledge, the works for BHP on the spherical coordinates are rarely investigated and even we do not search any results which is associated with the asymmetric and nonlinear case of the BHP. Based on the idea of two problems (1.5)-(1.7), we are interested in the nonlinear backward heat problem in spherical coordinates of determining the temperature distribution $u(r, \theta, \phi, t)$, $(r, \theta, \phi, t) \in (0, a) \times (0, \pi) \times (0, 2\pi) \times (0, T)$ satisfying the problem (1.1)-(1.4). It is easy to see that the problem (1.1)-(1.4) will be considered with the asymmetric case of the given data f and the nonlinear heat source q . To overcome the ill-posedness of the problem (1.1)-(1.4), we apply the modified quasi-boundary method (MQBV) to construct stable approximation problems and achieve the error estimates between the exact solution and the regularized solution. As we know, the quasi-boundary value (QBV) method, given by R. E. Showalter in 1983, is one of effective regularization methods. In [9], the main idea of the QBV method is to add an appropriate "corrector term" into the boundary condition. Based on this idea, the correction term was modified in [7], leading to the modified quasi-boundary value method. The MQBV method has been frequently applied by many authors in [2, 7, 12] in order to investigate to the BHP.

This paper is organized as follows: In Section 2, some definitions and propositions are given to find out the solution of the problem (1.1)-(1.4). In Section 3, the regularized solution by using the MQBV method and the proves of main results are presented.

2. Some Definitions and Propositions

Throughout this paper, we denote by $L^2(\Omega; r^2 \sin \theta)$ the space of Lebesgue measurable functions f with weight $r^2 \sin \theta$ on Ω .

Definition 2.1 Let $a > 0$, we define

$$H := L^2(\Omega; r^2 \sin \theta) = \left\{ v: \Omega \rightarrow \mathbb{C} \mid v \text{ is measurable, } \iiint_{\Omega} |v(r, \theta, \phi)|^2 r^2 \sin \theta d\phi d\theta dr < \infty \right\}.$$

The inner product and the corresponding norm in H are defined by

$$\langle v, w \rangle_H = \int_0^a \int_0^\pi \int_0^{2\pi} v(r, \theta, \phi) \overline{w(r, \theta, \phi)} r^2 \sin \theta d\phi d\theta dr,$$

and

$$\|v\|_H = \left(\int_0^a \int_0^\pi \int_0^{2\pi} |v(r, \theta, \phi)|^2 r^2 \sin \theta d\phi d\theta dr \right)^{\frac{1}{2}}.$$

From here on, some propositions which were presented in [4, 5, 16] are restated.

Definition 2.2 Let n be a non-negative integer. The spherical Bessel function of the first kind of order n is defined, for $x > 0$, by

$$j_n(x) = \left(\frac{\pi}{2x} \right)^{1/2} J_{n+\frac{1}{2}}(x),$$

where $J_{n+\frac{1}{2}}$ denotes the Bessel functions of the first kind of order $n+\frac{1}{2}$. At $x=0$, the function j_n is defined by continuous extension, namely, $j_0(0)=1$ and $j_n(0)=0, n=1,2,3,\dots$

Proposition 2.3 Let n be a non-negative integer. For $\lambda > 0$, consider the radial boundary-value problem

$$\begin{cases} r^2 R''(r) + 2rR'(r) + (\lambda^2 r^2 - n(n+1))R(r) = 0, & 0 < r < a, R(a) = 0, \\ R(r) \text{ is bounded as } r \rightarrow 0^+. \end{cases} \quad (2.1)$$

This problem admits a nontrivial regular solution if and only if

$$\lambda = \lambda_{n,j} = \frac{\alpha_{n+1/2,j}}{a}, \quad j = 1, 2, \dots,$$

where $\alpha_{n+1/2,j}$ is the j -th positive zero of $J_{n+\frac{1}{2}}$. The radial eigenfunctions corresponding to the eigenvalues $\lambda_{n,j}^2$ may be chosen as $R_{n,j}(r) = j_n(\lambda_{n,j}r)$, $j = 1, 2, \dots$. Equivalently, $j_n(\lambda_{n,j}a) = 0$. For fixed n , the family $\{j_n(\lambda_{n,j}r)\}_{j \in \mathbb{N}^*}$ satisfies the orthogonality relations

$$\int_0^a r^2 j_n(\lambda_{n,j}r) j_n(\lambda_{n,k}r) dr = 0, \quad j \neq k.$$

Moreover,

$$\int_0^a r^2 j_n^2(\lambda_{n,j}r) dr = \frac{a^3}{2} j_{n+1}^2(\alpha_{n+1/2,j}).$$

Proposition 2.4 Let n be a non-negative integer. The Legendre polynomial of degree n is given by

$$P_n(x) = \frac{1}{2^n} \sum_{m=0}^M (-1)^m \frac{(2n-2m)!}{m!(n-m)!(n-2m)!} x^{n-2m}, \quad (2.2)$$

where $M = n/2$ if n is even or $(n-1)/2$ if n is odd. Moreover, for $x \in (-1, 1)$, the Legendre function of the second kind Q_n admits the representation

$$Q_n(x) = \frac{1}{2} P_n(x) \ln\left(\frac{1+x}{1-x}\right) - \sum_{k=1}^n \frac{1}{k} P_{k-1}(x) P_{n-k}(x), \quad n = 0, 1, 2, \dots, \quad (2.3)$$

where the sum is understood to be zero when $n = 0$.

Proposition 2.5 Let n be a non-negative integer. The Legendre differential equation of degree n is

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0, \quad -1 < x < 1. \quad (2.4)$$

From that, the general solution of the equation (2.4) is

$$y(x) = c_1 P_n(x) + c_2 Q_n(x),$$

where $P_n(x)$, $Q_n(x)$ are defined by (2.2), (2.3) respectively, and c_1 , c_2 are arbitrary constants.

Remark 2.6

1. For $0 \leq m \leq n$, the associated Legendre function of degree n and order m is defined by

$$P_n^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m P_n(x)}{dx^m}, \quad x \in [-1, 1]. \quad (2.5)$$

For negative orders, we use the convention

$$P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x), \quad 0 \leq m \leq n.$$

2. The normalized complex spherical harmonics are defined by

$$Y_{n,m}(\theta, \phi) = \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} P_n^m(\cos \theta) e^{im\phi}, \quad (2.6)$$

where $n = 0, 1, 2, \dots$ and $m = -n, -(n-1), \dots, n-1, n$.

Proposition 2.7 Let n be a non-negative integer and consider the differential equation for the spherical harmonics be defined by

$$\frac{\partial^2 Y}{\partial \theta^2} + \cot \theta \frac{\partial Y}{\partial \theta} + \csc^2 \theta \frac{\partial^2 Y}{\partial \phi^2} + n(n+1)Y = 0,$$

where $0 < \theta < \pi, 0 < \phi < 2\pi$. Then $2n+1$ linearly independent nontrivial solutions are given by the spherical harmonics $Y(\theta, \phi) = Y_{n,m}(\theta, \phi)$, $|m| \leq n$, where $Y_{n,m}(\theta, \phi)$ is defined by (2.6).

Proposition 2.8 Let $f \in H$. Then the family

$$\{j_n(\lambda_{n,j}r)Y_{n,m}(\theta, \phi) : j \in \mathbb{N}^*, n \in \mathbb{N}, -n \leq m \leq n\}$$

forms a complete orthogonal system in H , and we have

$$f(r, \theta, \phi) = \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=-n}^n A_{jmn} j_n(\lambda_{n,j} r) Y_{n,m}(\theta, \phi),$$

with convergence in H , where

$$A_{jmn} = \frac{2}{a^3 j_{n+1}^2 \left(\alpha_{n+\frac{1}{2}, j} \right)} \int_0^a \int_0^\pi \int_0^{2\pi} f(r, \theta, \phi) j_n(\lambda_{n,j} r) \overline{Y_{n,m}(\theta, \phi)} r^2 \sin \theta d\phi d\theta dr,$$

and $\overline{Y_{n,m}}$ is the complex conjugate of $Y_{n,m}$. Furthermore, Parseval's identity takes the form

$$\|f\|_H^2 = \frac{a^3}{2} \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=-n}^n j_{n+1}^2 \left(\alpha_{n+\frac{1}{2}, j} \right) |A_{jmn}|^2.$$

Remark 2.9 If $f \in H$ is real-valued then its Fourier–Bessel coefficients satisfy $A_{j(-m)n} = (-1)^m \overline{A_{jmn}}$. Hence, the Fourier–Bessel expansion and the real-valued spectral filters used in Section 3 preserve real-valuedness.

3. Regularization and main results

Employing separation of variables, we can clearly find out the exact solution u of the problem (1.1)-(1.4) corresponding to the exact final data f as follows

$$u(r, \theta, \phi, t) = \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=-n}^n A_{jmn} [f, u](t) j_n(\lambda_{n,j} r) Y_{n,m}(\theta, \phi), \quad (3.1)$$

where

$$A_{jmn} [f, u](t) = \exp\{c^2 \lambda_{n,j}^2 (T-t)\} f_{jmn} - \int_t^T \exp\{c^2 \lambda_{n,j}^2 (s-t)\} q_{jmn}(u)(s) ds, \quad (3.2)$$

$$f_{jmn} = \frac{2}{a^3 j_{n+1}^2 \left(\alpha_{n+\frac{1}{2}, j} \right)} \int_0^a \int_0^\pi \int_0^{2\pi} f(r, \theta, \phi) j_n(\lambda_{n,j} r) \overline{Y_{n,m}(\theta, \phi)} r^2 \sin \theta d\phi d\theta dr,$$

$$q_{jmn}(u)(t) = \frac{2}{a^3 j_{n+1}^2 \left(\alpha_{n+\frac{1}{2}, j} \right)} \int_0^a \int_0^\pi \int_0^{2\pi} q(u)(r, \theta, \phi, t) j_n(\lambda_{n,j} r) \overline{Y_{n,m}(\theta, \phi)} r^2 \sin \theta d\phi d\theta dr.$$

Here j_n , $Y_{n,m}$, and $\lambda_{n,j}$ are defined in Definition 2.2, (2.6), and Proposition 2.3, respectively.

Since $0 \leq t < s \leq T$, we see from (3.2) that, as n becomes large, the terms

$$\exp\{c^2 \lambda_{n,j}^2 (T-t)\}, \exp\{c^2 \lambda_{n,j}^2 (s-t)\},$$

rise rather quickly. It is well known that the ill-posedness of problem (1.1)–(1.4) originates from these terms. To overcome this difficulty, one has to replace them with suitable stabilization terms. In detail, we replace $\exp\{c^2\lambda_{n,j}^2(T-t)\}, \exp\{c^2\lambda_{n,j}^2(s-t)\}$ by

$$\frac{\exp\{-c^2\lambda_{n,j}^2t\}}{\varepsilon c^2\lambda_{n,j}^2 + \exp\{-c^2\lambda_{n,j}^2T\}}, \frac{\exp\{c^2\lambda_{n,j}^2(s-t-T)\}}{\varepsilon c^2\lambda_{n,j}^2 + \exp\{-c^2\lambda_{n,j}^2T\}}, \text{ respectively. So that, we construct the}$$

regularized solutions $u^\varepsilon, v^\varepsilon$ corresponding the mesured data f^ε and the exact data f , respectively which are defined by

$$u^\varepsilon(r, \theta, \phi, t) = \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=-n}^n B_{jmn}^\varepsilon [f^\varepsilon, u^\varepsilon](t) j_n(\lambda_{n,j}r) Y_{n,m}(\theta, \phi), \quad (3.3)$$

where

$$B_{jmn}^\varepsilon [f^\varepsilon, u^\varepsilon](t) = \frac{\exp\{-c^2\lambda_{n,j}^2t\}}{\varepsilon c^2\lambda_{n,j}^2 + \exp\{-c^2\lambda_{n,j}^2T\}} f_{jmn}^\varepsilon - \int_t^T \frac{\exp\{c^2\lambda_{n,j}^2(s-t-T)\}}{\varepsilon c^2\lambda_{n,j}^2 + \exp\{-c^2\lambda_{n,j}^2T\}} q_{jmn}(u^\varepsilon)(s) ds,$$

$$f_{jmn}^\varepsilon = \frac{2}{a^3 j_{n+1}^2(\alpha_{n+1/2,j})} \int_0^a \int_0^\pi \int_0^{2\pi} f^\varepsilon(r, \theta, \phi) j_n(\lambda_{n,j}r) \overline{Y_{n,m}(\theta, \phi)} r^2 \sin \theta d\phi d\theta dr,$$

$$q_{jmn}(u^\varepsilon)(t) = \frac{2}{a^3 j_{n+1}^2(\alpha_{n+1/2,j})} \int_0^a \int_0^\pi \int_0^{2\pi} q(u^\varepsilon)(r, \theta, \phi, t) j_n(\lambda_{n,j}r) \overline{Y_{n,m}(\theta, \phi)} r^2 \sin \theta d\phi d\theta dr,$$

and

$$v^\varepsilon(r, \theta, \phi, t) = \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=-n}^n B_{jmn}^\varepsilon [f, v^\varepsilon](t) j_n(\lambda_{n,j}r) Y_{n,m}(\theta, \phi), \quad (3.4)$$

where

$$B_{jmn}^\varepsilon [f, v^\varepsilon](t) = \frac{\exp\{-c^2\lambda_{n,j}^2t\}}{\varepsilon c^2\lambda_{n,j}^2 + \exp\{-c^2\lambda_{n,j}^2T\}} f_{jmn} - \int_t^T \frac{\exp\{c^2\lambda_{n,j}^2(s-t-T)\}}{\varepsilon c^2\lambda_{n,j}^2 + \exp\{-c^2\lambda_{n,j}^2T\}} q_{jmn}(v^\varepsilon)(s) ds,$$

$$q_{jmn}(v^\varepsilon)(t) = \frac{2}{a^3 j_{n+1}^2(\alpha_{n+1/2,j})} \int_0^a \int_0^\pi \int_0^{2\pi} q(v^\varepsilon)(r, \theta, \phi, t) j_n(\lambda_{n,j}r) \overline{Y_{n,m}(\theta, \phi)} r^2 \sin \theta d\phi d\theta dr.$$

Here, ε is regularization parameter. Since the filter factors are real-valued and independent of m , Remark 2.9 shows that u^ε and v^ε are real-valued whenever the corresponding final data are real-valued.

Lemma 3.1 *Let $0 < \varepsilon < T, a > 0$, we have the following inequality*

$$\frac{1}{\varepsilon a + \exp\{-aT\}} \leq \frac{T}{\varepsilon} \left(\ln \left(\frac{T}{\varepsilon} \right) \right)^{-1}.$$

Lemma 3.2 *Let $0 \leq t \leq s \leq T, 0 < \varepsilon < T$ and $a > 0$, then we get the following inequalities*

$$1. \frac{\exp\{(s-t-T)a\}}{\varepsilon a + \exp\{-aT\}} \leq \tilde{T} \left(\varepsilon \ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{t-s}{T}},$$

2. *Let $s = T$, we obtain*

$$\frac{\exp\{-ta\}}{\varepsilon a + \exp\{-aT\}} \leq \tilde{T} \left(\varepsilon \ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{t}{T}-1},$$

$$3. \frac{\exp\{(s-t-T)a\}}{\varepsilon a + \exp\{-aT\}} \leq B_\varepsilon,$$

where $T = \max\{1; T\}$ and $B_\varepsilon = \max\left\{\frac{1}{\varepsilon}, \frac{T}{\varepsilon}\right\}$.

We can see the proof of the above lemma in [7].

We now state the assumptions imposed on the nonlinear source q . In particular, q is real-valued and globally Lipschitz continuous in its last variable. Thus, there exists a constant $L_q > 0$, independent of r, θ, ϕ, t, u, v such that

$$|q(u)(r, \theta, \phi, t) - q(v)(r, \theta, \phi, t)| \leq L_q |u - v|. \quad (3.5)$$

Next, we get the existence and the uniqueness of the solution of problem (3.3) in the following theorem.

Theorem 3.1 *Let $f^\varepsilon \in H$ be real-valued, $\varepsilon \in (0; \min\{1, T\})$, and suppose that q satisfies (3.5). Then the problem (3.3) has a unique real-valued solution $u^\varepsilon \in C([0, T]; H)$.*

Proof. For a real-valued $u \in C([0, T]; H)$, we define

$$W(u)(r, \theta, \phi, t) = \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=-n}^n B_{jnm}^\varepsilon [f^\varepsilon, u](t) j_n(\lambda_{n,j} r) Y_{n,m}(\theta, \phi),$$

where

$$B_{jnm}^\varepsilon [f^\varepsilon, u](t) = \frac{\exp\{-c^2 \lambda_{n,j}^2 t\}}{\varepsilon c^2 \lambda_{n,j}^2 + \exp\{-c^2 \lambda_{n,j}^2 T\}} f_{jnm} - \int_t^T \frac{\exp\{c^2 \lambda_{n,j}^2 (s-t-T)\}}{\varepsilon c^2 \lambda_{n,j}^2 + \exp\{-c^2 \lambda_{n,j}^2 T\}} q_{jnm}(u)(s) ds.$$

By Proposition 2.8, Remark 2.9, and the continuity assumptions on q , W maps the real-valued elements of $C([0, T]; H)$ into themselves. We equip $C([0, T]; H)$ with the norm

$$\|u\| = \sup_{t \in [0, T]} \|u(t)\|_H.$$

We claim that for every $u, v \in C([0, T]; H)$, k is a positive integer, we obtain

$$\|W^k(u)(t) - W^k(v)(t)\|_H^2 \leq \frac{(T-t)^k}{k!} T^k B_\varepsilon^{2k} L_q^{2k} \|u - v\|^2, \quad (3.6)$$

where $B_\varepsilon = \max \left\{ \frac{1}{\varepsilon}, \frac{T}{\varepsilon} \right\}$. We shall prove (3.6) by mathematical induction.

For $k = 1$, we get

$$\begin{aligned} \|W(u)(t) - W(v)(t)\|_H^2 &= \frac{a^3}{2} \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=-n}^n j_{n+1}^2 \left(\alpha_{n+\frac{1}{2}, j} \right) \\ &\quad \times \left| \int_t^T \frac{\exp\{c^2 \lambda_{n,j}^2 (s-t-T)\}}{\varepsilon c^2 \lambda_{n,j}^2 + \exp\{-c^2 \lambda_{n,j}^2 T\}} [q_{jmn}(u)(s) - q_{jmn}(v)(s)] ds \right|^2. \end{aligned}$$

By applying the Hölder inequality, Lemma 3.2 and (3.5), we have

$$\begin{aligned} &\|W(u)(t) - W(v)(t)\|_H^2 \\ &\leq (T-t) \frac{a^3}{2} \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=-n}^n j_{n+1}^2 \left(\alpha_{n+\frac{1}{2}, j} \right) \\ &\quad \times \int_t^T \left[\frac{\exp\{c^2 \lambda_{n,j}^2 (s-t-T)\}}{\varepsilon c^2 \lambda_{n,j}^2 + \exp\{-c^2 \lambda_{n,j}^2 T\}} \right]^2 |q_{jmn}(u)(s) - q_{jmn}(v)(s)|^2 ds \\ &\leq (T-t) \frac{a^3}{2} \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=-n}^n j_{n+1}^2 \left(\alpha_{n+\frac{1}{2}, j} \right) \\ &\quad \times \int_t^T \left(\tilde{T} \left(\varepsilon \ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{t-s}{T}} \right)^2 |q_{jmn}(u)(s) - q_{jmn}(v)(s)|^2 ds \\ &\leq TB_\varepsilon^2 \int_t^T \left[\frac{a^3}{2} \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=-n}^n j_{n+1}^2 \left(\alpha_{n+\frac{1}{2}, j} \right) |q_{jmn}(u)(s) - q_{jmn}(v)(s)|^2 \right] ds \\ &= TB_\varepsilon^2 \int_t^T \|q(u)(s) - q(v)(s)\|_H^2 ds \end{aligned}$$

$$\begin{aligned}
 &\leq TB_\varepsilon^2 L_q^2 \int_t^T \|u(s) - v(s)\|_H^2 ds \\
 &\leq TB_\varepsilon^2 L_q^2 \int_t^T \|u - v\|^2 ds \\
 &\leq T(T-t) B_\varepsilon^2 L_q^2 \|u - v\|^2.
 \end{aligned}$$

Thus, (3.6) holds for $k = 1$. Suppose that (3.6) holds for $k = i$, we prove that (3.6) holds for $k = i + 1$. Since $W^{i+1} = W \circ W^i$, the estimate established in the case $k = 1$, applied to $W^i(u)$ and $W^i(v)$, gives

$$\begin{aligned}
 \|W^{i+1}(u)(t) - W^{i+1}(v)(t)\|_H^2 &= \|W(W^i(u))(t) - W(W^i(v))(t)\|_H^2 \\
 &\leq TB_\varepsilon^2 L_q^2 \int_t^T \|W^i(u)(s) - W^i(v)(s)\|_H^2 ds.
 \end{aligned}$$

Using the induction hypothesis, we obtain

$$\begin{aligned}
 \|W^{i+1}(u)(t) - W^{i+1}(v)(t)\|_H^2 &= \|W(W^i(u))(t) - W(W^i(v))(t)\|_H^2 \\
 &\leq TB_\varepsilon^2 L_q^2 \int_t^T \frac{(T-s)^i}{i!} T^i B_\varepsilon^{2i} L_q^{2i} \|u - v\|^2 ds \\
 &= \frac{T^{i+1} B_\varepsilon^{2(i+1)} L_q^{2(i+1)}}{i!} \|u - v\|^2 \int_t^T (T-s)^i ds \\
 &= \frac{(T-t)^{i+1}}{(i+1)!} T^{i+1} B_\varepsilon^{2(i+1)} L_q^{2(i+1)} \|u - v\|^2.
 \end{aligned}$$

Hence, (3.6) holds for $k = i + 1$. Therefore, by mathematical induction, (3.6) holds for every positive integer k . Since $T - t \leq T$, we obtain

$$\|W^k(u) - W^k(v)\| \leq \frac{T^k}{\sqrt{k!}} B_\varepsilon^k L_q^k \|u - v\|.$$

For each fixed $\varepsilon > 0$,

$$\frac{T^k}{\sqrt{k!}} B_\varepsilon^k L_q^k \rightarrow 0,$$

as $k \rightarrow \infty$. Hence, there exists a positive integer number k_0 such that

$$\frac{T^{k_0}}{\sqrt{k_0!}} B_\varepsilon^{k_0} L_q^{k_0} < 1,$$

and W^{k_0} is a contraction. We conclude that the equation $W(u^\varepsilon) = u^\varepsilon$ has a unique solution $u^\varepsilon \in C([0, T]; H)$

The proof of Theorem 3.1 is completed.

In the following theorem, we shall prove the stability of the regularization method for problem (3.1).

Theorem 3.2 *Let $0 < \varepsilon < \min\{1, T\}$, and let $f, f^\varepsilon \in H$ be real-valued. Suppose that $u^\varepsilon, v^\varepsilon$ are defined by (3.3) and (3.4), corresponding to the final data f^ε and f , respectively. Then, we have*

$$\|u^\varepsilon(t) - v^\varepsilon(t)\|_H \leq \sqrt{2}\tilde{T} \left(\varepsilon \ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{t-T}{T}} \exp\{(T-t)^2 \tilde{T}^2 L_q^2\} \|f^\varepsilon - f\|_H. \quad (3.7)$$

Proof. From (3.3), (3.4), Proposition 2.8, Parseval's identity, Lemma 3.2 and

$$|a+b|^2 \leq 2|a|^2 + 2|b|^2,$$

we obtain

$$\begin{aligned} & \|u^\varepsilon(t) - v^\varepsilon(t)\|_H^2 \\ & \leq \frac{a^3}{2} \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=-n}^n 2j_{n+1}^2 \left(\alpha_{n+\frac{1}{2}, j} \right) \left| \frac{\exp\{-c^2 \lambda_{n,j}^2 t\}}{\varepsilon c^2 \lambda_{n,j}^2 + \exp\{-c^2 \lambda_{n,j}^2 T\}} (f_{jmn}^\varepsilon - f_{jmn}) \right|^2 \\ & + \frac{a^3}{2} \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=-n}^n 2j_{n+1}^2 \left(\alpha_{n+\frac{1}{2}, j} \right) \left| \int_t^T \frac{\exp\{c^2 \lambda_{n,j}^2 (s-t-T)\}}{\varepsilon c^2 \lambda_{n,j}^2 + \exp\{-c^2 \lambda_{n,j}^2 T\}} [q_{jmn}(u^\varepsilon)(s) - q_{jmn}(v^\varepsilon)(s)] ds \right|^2 \\ & \leq 2\tilde{T}^2 \left(\varepsilon \ln \left(\frac{T}{\varepsilon} \right) \right)^{2\left(\frac{t}{T}-1\right)} \|f^\varepsilon - f\|_H^2 \\ & + 2(T-t)\tilde{T}^2 \int_t^T \left(\varepsilon \ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{2(t-s)}{T}} \|q(u^\varepsilon)(s) - q(v^\varepsilon)(s)\|_H^2 ds \\ & \leq 2\tilde{T}^2 \left(\varepsilon \ln \left(\frac{T}{\varepsilon} \right) \right)^{2\left(\frac{t}{T}-1\right)} \|f^\varepsilon - f\|_H^2 \\ & + 2(T-t)\tilde{T}^2 L_q^2 \int_t^T \left(\varepsilon \ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{2(t-s)}{T}} \|u^\varepsilon(s) - v^\varepsilon(s)\|_H^2 ds \end{aligned}$$

It follows that

$$\begin{aligned} \left(\varepsilon \ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{-2t}{T}} \|u^\varepsilon(t) - v^\varepsilon(t)\|_H^2 &\leq 2T^2 \left(\varepsilon \ln \left(\frac{T}{\varepsilon} \right) \right)^{-2} \|f^\varepsilon - f\|_H^2 \\ &+ 2(T-t) \tilde{T}^2 L_q^2 \int_t^T \left(\varepsilon \ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{-2s}{T}} \|u^\varepsilon(s) - v^\varepsilon(s)\|_H^2 ds. \end{aligned}$$

Using Gronwall's inequality, we have

$$\|u^\varepsilon(t) - v^\varepsilon(t)\|_H \leq \sqrt{2} \tilde{T} \left(\varepsilon \ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{t-T}{T}} \exp\{(T-t)^2 \tilde{T}^2 L_q^2\} \|f^\varepsilon - f\|_H.$$

This completes the proof of Theorem 3.2.

Then we give estimate between the regularized solution corresponding to exact data and the exact solution as follows

Theorem 3.3 *Let $0 < \varepsilon < \min\{1, T\}$, and let v^ε be defined by (3.4), corresponding to the exact final data f . Let u be the exact solution of the problem (1.1)-(1.4). Assume that*

$$\|\nabla^2 u(\cdot, \cdot, \cdot, 0)\|_H \leq A, \quad (3.8)$$

and

$$\frac{a^3}{2} \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=-n}^n j_{n+1}^2 \left(\alpha_{n+\frac{1}{2}, j} \right) \int_0^T |c^2 \lambda_{n,j}^2 \exp\{c^2 \lambda_{n,j}^2 s\} q_{jmn}(u)(s)|^2 ds \leq B^2. \quad (3.9)$$

Then, for $t \in [0, T]$,

$$\begin{aligned} \|v^\varepsilon(t) - u(t)\|_H &\leq \sqrt{3} \tilde{T} (c^2 A + \sqrt{T} B) \exp\left\{\frac{3}{2}(T-t)^2 \tilde{T}^2 L_q^2\right\} \varepsilon^{\frac{t}{T}} \left(\ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{t-T}{T}}. \end{aligned} \quad (3.10)$$

Proof. From (3.2), evaluated at $t = 0$, we have

$$A_{jmn}[f, u](0) = \exp\{c^2 \lambda_{n,j}^2 T\} f_{jmn} - \int_0^T \exp\{c^2 \lambda_{n,j}^2 s\} q_{jmn}(u)(s) ds.$$

Subtracting $B_{jmn}^\varepsilon[f, v^\varepsilon](t)$ from $A_{jmn}[f, u](t)$, adding and subtracting $q_{jmn}(u)(s)$, and using

$$\begin{aligned} &\int_0^T c^2 \lambda_{n,j}^2 \exp\{c^2 \lambda_{n,j}^2 s\} q_{jmn}(u)(s) ds \\ &= \int_0^t c^2 \lambda_{n,j}^2 \exp\{c^2 \lambda_{n,j}^2 s\} q_{jmn}(u)(s) ds + \int_t^T c^2 \lambda_{n,j}^2 \exp\{c^2 \lambda_{n,j}^2 s\} q_{jmn}(u)(s) ds, \end{aligned}$$

we obtain

$$\begin{aligned}
 & A_{jmn} [f, u](t) - B_{jmn}^{\varepsilon} [f, v^{\varepsilon}](t) \\
 &= \frac{\varepsilon \exp\{-c^2 \lambda_{n,j}^2 t\}}{\varepsilon c^2 \lambda_{n,j}^2 + \exp\{-c^2 \lambda_{n,j}^2 T\}} \left(c^2 \lambda_{n,j}^2 \exp\{c^2 \lambda_{n,j}^2 T\} f_{jmn} - \int_0^T c^2 \lambda_{n,j}^2 \exp\{c^2 \lambda_{n,j}^2 s\} q_{jmn}(u)(s) ds \right) \\
 &+ \frac{\varepsilon \exp\{-c^2 \lambda_{n,j}^2 t\}}{\varepsilon c^2 \lambda_{n,j}^2 + \exp\{-c^2 \lambda_{n,j}^2 T\}} \int_0^t c^2 \lambda_{n,j}^2 \exp\{c^2 \lambda_{n,j}^2 s\} q_{jmn}(u)(s) ds \\
 &+ \int_t^T \frac{\exp\{c^2 \lambda_{n,j}^2 (s-t-T)\}}{\varepsilon c^2 \lambda_{n,j}^2 + \exp\{-c^2 \lambda_{n,j}^2 T\}} (q_{jmn}(v^{\varepsilon})(s) - q_{jmn}(u)(s)) ds. \tag{3.11}
 \end{aligned}$$

The expression in the first pair of square brackets satisfies

$$c^2 \lambda_{n,j}^2 \exp\{c^2 \lambda_{n,j}^2 T\} f_{jmn} - \int_0^T c^2 \lambda_{n,j}^2 \exp\{c^2 \lambda_{n,j}^2 s\} q_{jmn}(u)(s) ds = c^2 \lambda_{n,j}^2 A_{jmn} [f, u](0).$$

Since

$$\nabla^2 [j_n(\lambda_{n,j} r) Y_{n,m}(\theta, \phi)] = -\lambda_{n,j}^2 j_n(\lambda_{n,j} r) Y_{n,m}(\theta, \phi),$$

the quantity $c^2 \lambda_{n,j}^2 A_{jmn} [f, u](0)$ is the Fourier–Bessel coefficient of $-c^2 \nabla^2 u(\cdot, \cdot, 0)$.

By Proposition 2.8 and Parseval's identity,

$$\frac{a^3}{2} \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=-n}^n j_{n+1}^2 \left(\alpha_{n+\frac{1}{2}, j} \right) |c^2 \lambda_{n,j}^2 A_{jmn} [f, u](0)|^2 = c^4 \|\nabla^2 u(\cdot, \cdot, 0)\|_H^2.$$

Using assumption (3.8), we obtain

$$\frac{a^3}{2} \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=-n}^n j_{n+1}^2 \left(\alpha_{n+\frac{1}{2}, j} \right) |c^2 \lambda_{n,j}^2 A_{jmn} [f, u](0)|^2 \leq c^4 A^2.$$

By Proposition 2.8,

$$\|v^{\varepsilon}(t) - u(t)\|_H^2 = \frac{a^3}{2} \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=-n}^n j_{n+1}^2 \left(\alpha_{n+\frac{1}{2}, j} \right) |B_{jmn}^{\varepsilon} [f, v^{\varepsilon}](t) - A_{jmn} [f, u](t)|^2.$$

Using the inequality $|a+b+c|^2 \leq 3(|a|^2 + |b|^2 + |c|^2)$, we estimate the three terms in (3.11) separately.

Estimate of the first term:

By Lemma 3.2, with $s = T$, $\frac{\varepsilon \exp\{-c^2 \lambda_{n,j}^2 t\}}{\varepsilon c^2 \lambda_{n,j}^2 + \exp\{-c^2 \lambda_{n,j}^2 T\}} \leq \varepsilon \tilde{T} \left(\varepsilon \ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{t}{T}-1}$.

Combining this inequality with (3.8), the contribution of the first term is bounded above by $\varepsilon^2 \tilde{T}^2 \left(\varepsilon \ln \left(\frac{T}{\varepsilon} \right) \right)^{2\left(\frac{t}{T}-1\right)} c^4 A^2$.

Estimate of the second term:

By the Hölder inequality,

$$\begin{aligned} \left| \int_0^t c^2 \lambda_{n,j}^2 \exp\{c^2 \lambda_{n,j}^2 s\} q_{jmn}(u)(s) ds \right|^2 &\leq t \int_0^t \left| c^2 \lambda_{n,j}^2 \exp\{c^2 \lambda_{n,j}^2 s\} q_{jmn}(u)(s) \right|^2 ds \\ &\leq T \int_0^T \left| c^2 \lambda_{n,j}^2 \exp\{c^2 \lambda_{n,j}^2 s\} q_{jmn}(u)(s) \right|^2 ds. \end{aligned}$$

Using (3.9), the contribution of the second term is bounded above by

$$\varepsilon^2 \tilde{T}^2 \left(\varepsilon \ln \left(\frac{T}{\varepsilon} \right) \right)^{2\left(\frac{t}{T}-1\right)} T B^2.$$

Estimate of the third term:

By Lemma 3.2, for $t \leq s \leq T$, $\frac{\exp\{c^2 \lambda_{n,j}^2 (s-t-T)\}}{\varepsilon c^2 \lambda_{n,j}^2 + \exp\{-c^2 \lambda_{n,j}^2 T\}} \leq \tilde{T} \left(\varepsilon \ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{t-s}{T}}$.

By the Hölder inequality,

$$\begin{aligned} \left| \int_t^T \frac{\exp\{c^2 \lambda_{n,j}^2 (s-t-T)\}}{\varepsilon c^2 \lambda_{n,j}^2 + \exp\{-c^2 \lambda_{n,j}^2 T\}} (q_{jmn}(v^\varepsilon)(s) - q_{jmn}(u)(s)) ds \right|^2 \\ \leq (T-t) \tilde{T}^2 \int_t^T \left(\varepsilon \ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{2(t-s)}{T}} |q_{jmn}(v^\varepsilon)(s) - q_{jmn}(u)(s)|^2 ds. \end{aligned}$$

By Proposition 2.8 and Parseval's identity,

$$\frac{a^3}{2} \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=-n}^n j_{n+1}^2 \left(\alpha_{n+\frac{1}{2},j} \right) |q_{jmn}(v^\varepsilon)(s) - q_{jmn}(u)(s)|^2 = \|q(v^\varepsilon)(s) - q(u)(s)\|_H^2.$$

It follows from the Lipschitz condition (3.5) that

$$\|q(v^\varepsilon)(s) - q(u)(s)\|_H^2 \leq L_q^2 \|v^\varepsilon(s) - u(s)\|_H^2.$$

Therefore, the contribution of the third term is bounded above by

$$(T-t)\tilde{T}^2 L_q^2 \int_t^T \left(\varepsilon \ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{2(t-s)}{T}} \|v^\varepsilon(s) - u(s)\|_H^2 ds.$$

Combining the preceding estimates, we obtain

$$\begin{aligned} \|v^\varepsilon(t) - u(t)\|_H^2 &\leq 3\varepsilon^2 \tilde{T}^2 \left(\varepsilon \ln \left(\frac{T}{\varepsilon} \right) \right)^{2\left(\frac{t}{T}-1\right)} (c^4 A^2 + TB^2) \\ &\quad + 3(T-t)\tilde{T}^2 L_q^2 \int_t^T \left(\varepsilon \ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{2(t-s)}{T}} \|v^\varepsilon(s) - u(s)\|_H^2 ds. \end{aligned}$$

Multiplying both sides by $\varepsilon^{\frac{-2t}{T}} \left(\ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{2T-2t}{T}}$, and noting that

$$\varepsilon^{\frac{-2t}{T}} \left(\ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{2T-2t}{T}} \varepsilon^2 \left(\varepsilon \ln \left(\frac{T}{\varepsilon} \right) \right)^{2\left(\frac{t}{T}-1\right)} = 1,$$

and, for $t \leq s \leq T$,

$$\varepsilon^{\frac{-2t}{T}} \left(\ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{2T-2t}{T}} \left(\varepsilon \ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{2(t-s)}{T}} = \varepsilon^{\frac{-2s}{T}} \left(\ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{2(T-s)}{T}},$$

we obtain

$$\begin{aligned} &\varepsilon^{\frac{-2t}{T}} \left(\ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{2T-2t}{T}} \|v^\varepsilon(t) - u(t)\|_H^2 \\ &\leq 3\tilde{T}^2 (c^4 A^2 + TB^2) + 3(T-t)\tilde{T}^2 L_q^2 \int_t^T \varepsilon^{\frac{-2s}{T}} \left(\ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{2T-2s}{T}} \|v^\varepsilon(s) - u(s)\|_H^2 ds. \end{aligned}$$

Using Gronwall's inequality, we have

$$\varepsilon^{\frac{-2t}{T}} \left(\ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{2T-2t}{T}} \|v^\varepsilon(t) - u(t)\|_H^2 \leq 3\tilde{T}^2 (c^4 A^2 + TB^2) \exp\{3(T-t)^2 \tilde{T}^2 L_q^2\}.$$

Taking square roots, we obtain

$$\begin{aligned} \|v^\varepsilon(t) - u(t)\|_H &\leq \sqrt{3T} \sqrt{c^4 A^2 + TB^2} \exp\left\{\frac{3}{2}(T-t)^2 T^2 L_q^2\right\} \varepsilon^{\frac{t}{T}} \left(\ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{t-T}{T}} \\ &\leq \sqrt{3T} (c^2 A + \sqrt{T} B) \exp\left\{\frac{3}{2}(T-t)^2 T^2 L_q^2\right\} \varepsilon^{\frac{t}{T}} \left(\ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{t-T}{T}}. \end{aligned}$$

This completes the proof of Theorem 3.3.

Finally, we prove the error estimate between the regularized solution corresponding to the measured data f^ε and the exact solution of the problem (1.1)-(1.4).

Theorem 3.4 Suppose that $f, f^\varepsilon \in H$ are real-valued and satisfy $\|f^\varepsilon - f\|_H \leq \varepsilon$.

Let u^ε and v^ε be defined by (3.3), (3.4) corresponding to the final data f^ε and f , respectively. Let u be the exact solution of the problem (1.1)-(1.4) satisfying (3.8) and (3.9).

If $\varepsilon \in (0; \min\{1, T\})$ then

$$\|u^\varepsilon(t) - u(t)\|_H \leq \varepsilon^{\frac{t}{T}} \left(\ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{t-T}{T}} M, \quad 0 \leq t \leq T,$$

where $M = \sqrt{2}\tilde{T} \exp\{\tilde{T}^4 L_q^2\} + \sqrt{3}\tilde{T}(c^2 A + \sqrt{T}B) \exp\left\{\frac{3}{2}\tilde{T}^4 L_q^2\right\}$.

Proof. By using triangle inequality, we infer that

$$\|u^\varepsilon(t) - u(t)\|_H \leq \|u^\varepsilon(t) - v^\varepsilon(t)\|_H + \|v^\varepsilon(t) - u(t)\|_H. \quad (3.12)$$

From (3.7), (3.10) and (3.12), we obtain

$$\begin{aligned} \|u^\varepsilon(t) - u(t)\|_H &\leq \sqrt{2}T\varepsilon^{\frac{t}{T}} \left(\ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{t-T}{T}} \exp\left\{(T-t)^2 T^2 L_q^2\right\} \\ &+ \sqrt{3}T(c^2 A + \sqrt{T}B) \exp\left\{\frac{3}{2}(T-t)^2 T^2 L_q^2\right\} \varepsilon^{\frac{t}{T}} \left(\ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{t-T}{T}} \\ &\leq \varepsilon^{\frac{t}{T}} \left(\ln \left(\frac{T}{\varepsilon} \right) \right)^{\frac{t-T}{T}} M, \end{aligned}$$

where $M = \sqrt{2}T \exp\left\{T^4 L_q^2\right\} + \sqrt{3}T(c^2 A + \sqrt{T}B) \exp\left\{\frac{3}{2}T^4 L_q^2\right\}$.

This completes the proof of Theorem 3.4.

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