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LOCAL POLYNOMIAL CONVEXITY OF UNION OF TOTALLY REAL GRAPHS OF FUNCTIONS IN TWO VARIABLES IN $\mathbb{C}^3$

Tính chất lồi đa thức địa phương của hợp các đồ thị hoàn toàn thực của các hàm hai biến số trong $\mathbb{C}^3$

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Abstract

In this note, we establish sufficient conditions under which the union of a totally real graph of a function in two variables and its tangent spaces is locally polynomially convex at the origin in $\mathbb{C}^3$. Furthermore, several examples are provided to demonstrate that the conditions we impose are both necessary and reasonable.

Keywords*: polynomially convex; totally real; Kallin's lemma.*

Tóm tắt

Trong bài báo này, chúng tôi thiết lập các điều kiện đủ để hợp của một đồ thị hoàn toàn thực của một hàm hai biến và các không gian tiếp xúc của nó là lồi đa thức địa phương tại gốc trong $\mathbb{C}^3$. Hơn nữa, chúng tôi đưa ra một số ví dụ nhằm minh họa rằng các điều kiện được đặt ra là cần thiết và hợp lý.

Từ khóa: *tính lồi đa thức; hoàn toàn thực; bổ đề Kallin.*

1. Introduction

Let K be a compact subset of $\mathbb{C}^n$. We say that K is polynomially convex if $\widehat{K} = K$, where $\widehat{K}$ denotes the polynomially convex hull of K, that is,

$$\widehat{K} = \{z \in \mathbb{C}^n : |p(z)| \leq \| p \|_K \quad \text{for every holomorphic polynomial } p \text{ on } \mathbb{C}^n\}.$$

A compact set K is said to be locally polynomially convex at a point $a \in K$ if there exists a closed ball $B(a)$ centered at a such that $B(a) \cap K$ is polynomially convex. The study of polynomial convexity is largely motivated by the celebrated Oka–Weil approximation theorem (see [1], p. 36), which asserts that every function holomorphic in a neighborhood of a compact polynomially convex set in $\mathbb{C}^n$ can be uniformly approximated by polynomials in $\mathbb{C}^n$. In one complex variable, a compact set $K \subset \mathbb{C}$ is polynomially convex if and only if its complement $\mathbb{C} \setminus K$ is connected. In higher dimensions, however, no such simple topological characterization exists, and determining whether a given compact set is polynomially convex is often a difficult task. By a classical theorem of

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Wermer ([18]; see also [1], Theorem 17.1), every totally real manifold is locally polynomially convex. Recall that a $\mathcal{C}^1$-smooth real manifold M is called totally real at a point $p \in M$ if its real tangent space T_pM contains no complex line. When the manifold in question is the union of two real planes, a complete characterization was provided by Weinstock. The present paper is devoted to the local polynomial convexity at the origin for a special class of sets in $\mathbb{C}^3$: the union of two totally real 3-dimensional submanifolds X_1 and X_2 that share the same tangent space at the origin. Our main result is strongly inspired by several previous studies on analogous situations in $\mathbb{C}^2$, which were motivated by questions of local approximation and carried out by De Paepe and Nguyen Quang Dieu. More precisely, we describe the following setup:

$$X_1 = \{(z_1, z_2, \overline{z_1} + \overline{z_2})\}$$

and

$$X_2 = \{\big(z_1, z_2, \overline{z_1} + \overline{z_2} + \varphi(z_1, z_2)\big)\},$$

where $\varphi(z_1, z_2)$ is of class $\mathcal{C}^1$ in a neighborhood of $(0,0)$ in $\mathbb{C}^2$ and $\varphi(z_1, z_2) = o\Big(\sqrt{|z_1|^2 + |z_2|^2}\Big)$. It is easy to see that X_1 and X_2 are totally real at $(0,0,0)$. By the Wermer–Hörmander theorem, they are locally polynomially convex at $(0,0,0)$. Our goal is to find a class of functions φ such that $X_1 \cup X_2$ is locally polynomially convex at $(0,0,0)$. We would like to emphasize that, in [16], the authors investigated the problem in the case where the functions are of a single variable, namely

$$X_1 := \{\big(z, f_1(z), f_2(z)\big): z \in U\},$$
$$X_2 := \{\big(z, f_3(z), f_4(z)\big): z \in U\},$$

where U is a small neighborhood of 0 in $\mathbb{C}$.

It should also be emphasized that the most difficult part of the argument is to prove the polynomial convexity of the union of several totally real graphs of functions of several complex variables. In contrast to the one-dimensional case, the higher-dimensional situation poses considerable difficulties for the application of Kallin's lemma, as the pairwise intersections of these graphs can be real submanifolds of dimension two.

2. Local polynomial convexity of union of totally real several variables graphs

As in most earlier studies on this topic ([3], [4], [5], [6], [11], [12], …), we shall repeatedly invoke a highly useful result of Kallin that provides a sufficient condition ensuring the polynomial convexity of the union of two polynomially convex sets.

Theorem 2.1. *(Kallin's lemma [9, 14]) Suppose that:*

1) X_1 and X_2 are polynomially convex subsets of $\mathbb{C}^n$;

2) Y_1 and Y_2 are polynomially convex subsets of $\mathbb{C}$ such that 0 is a boundary point of both Y_1 and Y_2 and $Y_1 \cap Y_2 = \{0\}$;

3) p is a polynomial such that $p(X_1) = Y_1$ and $p(X_2) = Y_2$;

4) $p^{-1}(0) \cap (X_1 \cup X_2)$ is polynomially convex.

Then $X_1 \cup X_2$ is polynomially convex.

It follows directly from Kallin's lemma that the union of two disjoint convex compact sets in $\mathbb{C}^n$ is always polynomially convex.

The following theorem constitutes the main result of this paper and is motivated by the known single-variable case.

Theorem 2.2. *Suppose that* $\varphi(z_1, z_2) = \sum_{k=0}^{m} a_k (z_1 + z_2)^k (\overline{z_1} + \overline{z_2})^{m-k} + o\left(\left(\sqrt{|z_1|^2 + |z_2|^2}\right)^m\right)$ $(m \geq 2)$ *is a $\mathcal{C}^1$ function in a neighborhood of* $(0,0)$ *in* $\mathbb{C}^2$ *and there exists* $l \geq \frac{m}{2}$ *such that* $|a_l| > \sum_{k \neq l} |a_k|$. (2.1) *Then $X_1 \cup X_2$ is locally polynomially convex at* $(0,0,0) \in \mathbb{C}^3$.

Proof. For each $r > 0$, we set

$$X_i^r = \{(z_1, z_2, w) \in X_i : |z_1| \leq r, |z_2| \leq r\}, \ i = 1,2.$$

Consider the polynomial

$$p(z, u, v) = \overline{\alpha}(z + u)^{m-2l+1} + \alpha v^{m-2l+1},$$

where α is suitably chosen later. We have

$$\begin{aligned} p(z_1, z_2, \overline{z_1} + \overline{z_2}) &= \overline{\alpha}(z_1 + z_2)^{m-2l+1} \\ &\quad + \alpha(\overline{z_1} + \overline{z_2})^{m-2l+1} \\ &= \overline{\alpha}(z_1 + z_2)^{m-2l+1} + \alpha\overline{z_1 + z_2}^{m-2l+1}. \end{aligned}$$

This implies that $p(X_1^r) \subset \mathbb{R}$. On the other hand, we have

$$\begin{aligned} p(z_1, z_2, w) &= \overline{\alpha}(z_1 + z_2)^{m-2l+1} \\ &\quad + \alpha(\overline{z_1} + \overline{z_2} + \sum_{k=0}^{m} a_k (z_1 + z_2)^{m-k} (\overline{z_1} + \overline{z_2})^k + o\left(\left(\sqrt{|z_1|^2 + |z_2|^2}\right)^m\right))^{m-2l+1} \\ &= \overline{\alpha}(z_1 + z_2)^{m-2l+1} + \alpha\overline{z_1 + z_2}^{m-2l+1} \\ &\quad + \alpha(m - 2l + 1)\overline{z_1 + z_2}^{m-2l} \sum_{k=0}^{m} a_k \left(\overline{z_1 + z_2}\right)^k (z_1 + z_2)^{m-k} \\ &\quad + o\left(\left(\sqrt{|z_1|^2 + |z_2|^2}\right)^m\right) \end{aligned}$$

for every $(z_1, z_2, w) \in X_2^r$. Since

$$\overline{\alpha}(z_1 + z_2)^{m-2l+1} + \alpha\overline{z_1 + z_2}^{m-2l+1} \in \mathbb{R}$$

for all z_1, z_2, it follows that

$$\mathrm{Im}p(z_1,z_2,w) = \mathrm{Im}(\alpha(m-2l+1)\big(\overline{z_1+z_2}\big)^{m-2l}\sum_{k=0}^{m} a_k\big(\overline{z_1+z_2}\big)^{k}(z_1+z_2)^{m-k} + o\left(\left(\sqrt{|z_1|^2+|z_2|^2}\right)^{m}\right)),$$

for all $(z_1, z_2, w) \in X_2^r$. If we fix $\alpha = i\frac{\overline{a_l}}{|a_l|}$, then

$$\begin{aligned}\mathrm{Im}p(z_1,z_2,w) &= \mathrm{Im}(m-2l+1)\left[\frac{i}{|a_l|}\left(|a_l|^2|z_1+z_2|^{2m-2l}+\sum_{k\neq l}\overline{a_l}\,a_k\overline{z_1+z_2}^{\,m-2l+k}(z_1+z_2)^{m-k}\right)+o\left(\left(\sqrt{|z_1|^2+|z_2|^2}\right)^{m}\right)\right] \\ &\geq |z_1+z_2|^{2m-2l}(m-2l+1)\left(|a_l|-\sum_{k\neq l}|a_k|\right)+o\left((\sqrt{|z_1|^2+|z_2|^2})^{m}\right),\end{aligned} \quad (2.2)$$

for all $(z_1, z_2, w) \in X_2^r$. Combining (2.1) and (2.2), we obtain $\mathrm{Im}p(z_1, z_2, w) > 0$ for every $(z_1, z_2, w) \in X_2^r$ when r is sufficiently small. Hence $p(X_2^r) \cap \mathbb{R} = \{0\}$. This implies that $p(X_1^r) \cap p(X_2^r) = \{0\}$ if r is sufficiently small. On the other hand, it follows from (2.2) that

$$p^{-1}(0) \cap X_2^r = \{(0,0,0)\} \in \mathbb{C}^3.$$

By a simple computation, we obtain

$$C := p^{-1}(0) \cap X_1^r = \{\big(z_1, z_2, \rho e^{-i\theta}\big): 0 \leq \rho \leq r\},$$

where θ is a real constant and $z_1 + z_2 = \rho e^{i\theta}, |z_1| \leq r, |z_2| \leq r$. We shall show that C is a convex set. Indeed, let $u = \big(z_1, z_2, \rho_1 e^{-i\theta}\big) \in C$ and $v = \big(z_3, z_4, \rho_2 e^{-i\theta}\big) \in C$ and $\lambda \in [0,1]$. We have

$$\lambda u + (1-\lambda)v = \big(\lambda z_1 + (1-\lambda)z_3, \lambda z_2 + (1-\lambda)z_4, \rho e^{-i\theta}\big),$$

and

$$\big(\lambda(z_1+z_2) + (1-\lambda)(z_3+z_4)\big) = (\lambda\rho_1 + (1-\lambda)\rho_2)e^{i\theta},$$

where

$$\rho = \lambda\rho_1 + (1-\lambda)\rho_2 \leq \lambda r + (1-\lambda)r = r.$$

Moreover, we have

$$|\lambda z_1 + (1-\lambda)z_3| \leq \lambda|z_1| + (1-\lambda)|z_3| \leq \lambda r + (1-\lambda)r = r$$

and

$$|\lambda z_2 + (1-\lambda)z_4| \leq \lambda|z_2| + (1-\lambda)|z_4| \leq \lambda r + (1-\lambda)r = r.$$

This yields $\lambda u + (1-\lambda)v \in C$. We can conclude that

$$p^{-1}(0) \cap (X_1 \cup X_2) = (p^{-1}(0) \cap X_1) \cup \{(0,0)\} = p^{-1}(0) \cap X_1 = C$$

is a convex set. Hence C is polynomially convex. By Kallin's lemma we deduce that $X_1^r \cup X_2^r$ is polynomially convex for all sufficiently small r.

We easily obtain the following corollaries.

Corollary 2.3. *Let* $\varphi(z_1, z_2) = \sum_{k=0}^{m} a_k (z_1)^k (\overline{z_1})^{m-k} + o\left(\left(\sqrt{|z_1|^2 + |z_2|^2}\right)^m\right) (m \geq 2)$ *be a function of class* $\mathcal{C}^1$ *in a neighborhood of* $(0,0)$ *in* $\mathbb{C}^2$. *Suppose that there exists* $l \geq \frac{m}{2}$ *such that* $|a_l| > \sum_{k \neq l} |a_k|$. *Then* $Y_1 \cup Y_2$ *is locally polynomially convex at* $(0,0,0)$, *where* $Y_1 = \{(z_1, z_2, \overline{z_1})\}$ *and* $Y_2 = \{\left(z_1, z_2, \overline{z_1} + \varphi(z_1, z_2)\right)\}$.

Corollary 2.4. *Let* $\varphi(z_1, z_2) = \sum_{k=0}^{m} a_k (z_2)^k (\overline{z_2})^{m-k} + o\left(\left(\sqrt{|z_1|^2 + |z_2|^2}\right)^m\right) (m \geq 2)$ *be a function of class* $\mathcal{C}^1$ *in a neighborhood of* $(0,0)$ *in* $\mathbb{C}^2$. *Suppose that there exists* $l \geq \frac{m}{2}$ *such that* $|a_l| > \sum_{k \neq l} |a_k|$. *Then* $Z_1 \cup Z_2$ *is locally polynomially convex at* $(0,0,0)$, *where* $Z_1 = \{(z_1, z_2, \overline{z_2})\}$ *and* $Z_2 = \{\left(z_1, z_2, \overline{z_2} + \varphi(z_1, z_2)\right)\}$.

The following theorem shows that the condition $l \geq \frac{m}{2}$ in Theorem 2.2 cannot be dropped.

Theorem 2.5. *Let* p *be a positive integer and* $X_1 = \{(z_1, z_2, \overline{z_1} + \overline{z_2}) : (z_1, z_2) \in D\}$, $X_2 = \{(z_1, z_2, \overline{z_1} + \overline{z_2} + (z_1 + z_2)^p (\overline{z_1} + \overline{z_2})^{1+p}) : (z_1, z_2) \in D\}$, *where* D *is a closed bidisk centered at* $(0,0) \in \mathbb{C}^2$. *Then* $X_1 \cup X_2$ *is not locally polynomially convex at* $(0,0,0) \in \mathbb{C}^3$.

Proof. We set $W_t = \{(z, w, u) : (z + w)u = t\}$ for each $t > 0$. Let

$$P_t := W_t \cap X_1 = \{(z, w, u) : |z| = t\}$$

and

$$Q_t := W_t \cap X_2 = \{(z, w, u) : |z| = s\},$$

where s is the unique solution of the equation

$$x^2 + x^{2(1+p)} = t.$$

By the maximum modulus principle, we infer that $X_1^r \cup X_2^r$ contains the component of W_t bounded by the closed curves P_t and Q_t for every t >. This implies that $X_1^r \cup X_2^r$ is not locally polynomially convex for any $r > 0$. The theorem is proved.

Next, we shall consider:

$$Y_1 = \{(z_1, z_2, \overline{z_1 z_2})\}$$

and

$$Y_2 = \{\left(z_1, z_2, \overline{z_1 z_2} + \varphi(z_1, z_2)\right)\},$$

where $\varphi(z_1, z_2)$ is of class $\mathcal{C}^1$ in a neighborhood of $(0,0)$ in $\mathbb{C}^2$ and $\varphi(z_1, z_2) = o(|z_1 z_2|)$. By the same argument as in the proof of Theorem 2.2, we obtain the following result. We omit the detailed proof.

Theorem 2.6. *Suppose that* $\varphi(z_1, z_2) = \sum_{k=0}^{m} a_k (z_1 z_2)^k (\overline{z_1 z_2})^{m-k} + o(|z_1 z_2|^m)$ $(m \geq 2)$ *is a* $\mathcal{C}^1$ *function in a neighborhood of* $(0,0)$ *in* $\mathbb{C}^2$ *and there exists* $l \geq \frac{m}{2}$ *such that* $|a_l| > \sum_{k \neq l} |a_k|$. *Then* $Y_1 \cup Y_2$ *is locally polynomially convex at* $(0,0,0) \in \mathbb{C}^3$.

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REFERENCES

H. Alexander and J. Wermer, *Several Complex Variables and Banach Algebras*, Grad. Texts in Math., **35**, Springer- Verlag, New York, 1998.

E. M. Chirka, *Approximation by holomorphic functions on smooth manifolds in* $\mathbb{C}^n$, Mat. Sb. (N.S.), **78** (1969), no. 10, 101-123.

Kieu Phuong Chi, *Local polynomial convexity of graphs of functions in several variables*, J. Math. Anal. Appl., **433** (2016), no. 1, 473-486.

Nguyen Quang Dieu, *Local polynomial convexity of tangential unions of totally real graphs in* $\mathbb{C}^2$, Indagationes Math., **10** (1999), no. 3, 349-355.

Nguyen Quang Dieu, *Local hull of perturbations of a union of totally real planes meeting along a real line*, Publ. Mat., **45** (2001), no. 2, 335-341.

Nguyen, Quang Dieu and P. De Paepe, *Function algebras on disks*, Complex Var. Theory Appl., **47** (2002), no. 5, 447-451.

S. Gorai, *A note on polynomial convexity of the union of finitely many totally-real planes in* $\mathbb{C}^2$, J. Math. Anal. Appl., **418** (2014), no. 2, 842-851.

L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, Third edition. North-Holland Mathematical Library, 7. North-Holland Publishing Co., Amsterdam, 1990. xii+254 pp.

E. Kallin, *Fat polynomially convex sets*, Function Algebras, (Proc. Inter. Symp. on Function Algebras, Tulane Univ, 1965, Scott Foresman, Chicago, 149-152 (1966).

A. G. O'Farrell, K.J. Preskenis and D. Walsh, *Holomorphic approximation in Lipschitz norms*, Contemp. Math., **32**(1984), 187-194.

P.J. de Paepe, *Approximation on a disk I*, Math. Zeit., **212**, 145-152 (1993).

P.J. de Paepe, *Eva Kallin's lemma on polynomial convexity*, Bull. of London Math. Soc., **33** (2001), 1-10.

K.J. Preskenis *Approximation on disks,* Tran. Amer. Math. Soc., **171** (1972**)**, 445-467.

E. L. Stout, *Polynomial convexity*, Progress in Mathematics, 261. Birkháuser Boston, 2007.

G. Stolzenberg, *Polynomially and rationally convex sets,* Acta Math., **109** (1963), 259-269.

M. T.Tan and K. P. Chi. *Local polynomial convexity of the union of two tangential totally real graphs in* $\mathbb{C}^3$, Acta Math. Vietnam., **50** (2025), no. 3, 411-418.

B.M. Weinstock, *On the polynomial convexity of the union of two maximal totally real subspaces of* $\mathbb{C}^n$, Math. Ann., **282** (1988), no. 1, 131-138.

J. Wermer, *Approximation on a disk,* Math Ann.,**155** (1964) 331-333.

J. Wiegerinck, *Locally polynomially convex hulls at degenerated CR singularities of surfaces in* $\mathbb{C}^2$, Indiana Univ. Math. J., **44** (1995), 897-915.

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