TRƯỜNG ĐẠI HỌC SÀI GÒN
**TẠP CHÍ KHOA HỌC**
**TRƯỜNG ĐẠI HỌC SÀI GÒN**
ISSN 1859-3208
e-ISSN 3093-3056
Số 102, Tập 03/2025

SAIGON UNIVERSITY
**SCIENTIFIC JOURNAL**
**OF SAIGON UNIVERSITY**
No. 102, Vol 03/2025

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# A BACKWARD PROBLEM FOR THE HOMOGENEOUS FRACTIONAL HELMHOLTZ EQUATION WITH GAUSSIAN WHITE NOISE MEASUREMENT

## *Bài toán ngược cho phương trình Helmholtz thuần nhất chứa đạo hàm bậc không nguyên với dữ liệu nhiễu trắng Gauss*

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**Abstract**

We investigate a backward final value problem for a homogeneous fractional Helmholtz equation on the strip $(0,\pi)\times[0,1]$. The final observations are assumed to be corrupted by Gaussian white noise. Since the solution operator contains exponentially increasing Fourier multipliers, the problem is severely ill-posed in the sense of Hadamard. A Fourier truncation method is proposed to stabilize the reconstruction. Under Sobolev-type smoothness assumptions on the exact final data, and under two different source conditions imposed on the exact solution, we derive convergence estimates for the expectation of the reconstruction error. In the $L^2(0,\pi)$ norm, logarithmic-type convergence is obtained under a weaker exponentially weighted condition, whereas a Hölder-type rate is established under a stronger exponential source condition. An additional estimate in the $H^q(0,\pi)$ norm is also proved. The results show that the truncation method provides a simple and effective regularization strategy for the homogeneous fractional Helmholtz backward problem with Gaussian white noise measurements. **2020 Mathematics Subject Classification.** 35J05, 35R11, 47A52, 60G15, 35R30.

***Keywords****: Homogeneous fractional Helmholtz equation, backward problem; Gaussian white noise; Fourier truncation method; regularization; ill-posed problem.*

**Tóm tắt**

Chúng tôi nghiên cứu bài toán giá trị cuối cho phương trình Helmholtz thuần nhất chứa đạo hàm bậc không nguyên trên miền $(0,\pi)\times[0,1]$. Các quan sát tại thời điểm cuối được giả thiết bị nhiễu bởi nhiễu trắng Gauss. Do toán tử nghiệm chứa các nhân tử Fourier tăng theo hàm mũ, bài toán là không chỉnh theo nghĩa Hadamard. Do đó, một phương pháp chặt cụt Fourier được đề xuất nhằm ổn định quá trình khôi phục nghiệm. Dưới các giả thiết về tính trơn kiểu Sobolev trên dữ liệu chính xác, cùng với hai điều kiện tiên nghiệm khác nhau trên nghiệm chính xác, chúng tôi thiết lập các ước lượng hội tụ cho kỳ vọng của sai số khôi phục. Trong không gian $L^2(0,\pi)$, ta thu được tốc độ hội tụ kiểu logarit dưới một điều kiện có trọng số mũ yếu hơn, trong khi tốc độ hội tụ kiểu Hölder được thiết lập dưới một điều kiện nguồn dạng mũ mạnh hơn. Ngoài ra, một ước lượng trong không gian $H^q(0,\pi)$ cũng được chứng minh. Các kết quả cho thấy phương pháp chặt cụt Fourier cung cấp một sơ đồ chỉnh hóa đơn giản và hiệu quả cho bài toán ngược theo thời gian của phương trình Helmholtz chứa đạo hàm bậc không nguyên thuần nhất, khi các phép đo bị nhiễu trắng Gauss.

***Từ khóa:*** *phương trình Helmholtz thuần nhất chứa đạo hàm bậc không nguyên; bài toán ngược; nhiễu trắng Gauss; phương pháp chặt cụt Fourier; chỉnh hóa; bài toán đặt không chỉnh.*

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## 1. Introduction

The Helmholtz equation is one of the standard models for the description of time-harmonic wave propagation. It appears in acoustics, electromagnetism, optics, seismic imaging, radar, sonar, and medical imaging [4, 6, 8]. In many applications, the direct problem consists of predicting a wave field from known boundary or initial information. In contrast, inverse and backward problems aim to reconstruct an unknown state of the system from indirect or terminal observations. Such problems are important but usually unstable: small errors in the measured data may be amplified dramatically by the reconstruction operator [3, 6, 11].

Fractional variants of classical elliptic and wave equations have attracted considerable attention in recent years. Fractional Laplacian operators are useful in the description of anomalous diffusion, long-range interactions, nonlocal media, rough structures, and wave propagation in complex environments [2, 4, 7]. In a bounded interval, the spectral fractional Laplacian provides a convenient and rigorous framework because it is naturally connected with the eigenfunctions of the Dirichlet Laplacian. This spectral structure is especially suitable for constructing regularized solutions by Fourier methods [1].

During the last two decades, numerous regularization methods have been proposed for inverse and backward problems involving parabolic, elliptic, and hyperbolic equations. These include Fourier truncation methods, quasi-boundary value methods, Tikhonov regularization, mollification techniques, and spectral filtering methods. Among these approaches, Fourier truncation is particularly attractive because of its conceptual simplicity and its compatibility with spectral representations of differential operators. A number of convergence analyses for backward heat equations and related inverse problems can be found in [3, 5, 6, 11].

In this work, we study the following final value problem for the homogeneous fractional Helmholtz equation:

$$\Delta u(x,y) - (-\Delta)^{\alpha} u(x,y) + k^2 u(x,y) = 0, (x,y) \in (0,\pi) \times [0,1], \quad (1)$$

subject to the boundary and final conditions

$$u(0,y) = u(\pi,y) = 0, y \in [0,1], \quad (2)$$

$$u(x,1) = g(x), x \in (0,\pi), \quad (3)$$

$$u_y(x,1) = h(x), x \in (0,\pi). \quad (4)$$

Here $\alpha \in (0,1)$, the number $k \in \left(0,\sqrt{2}\right)$ is given, and $g, h$ denote final measured quantities. The operator $(-\Delta)^{\alpha}$ is understood in the spectral sense with respect to the Dirichlet eigenfunctions on $(0,\pi)$.

The problem (1)-(4) is backward in the variable $y$. When the solution is expanded into a sine series, the Fourier coefficients are multiplied by the functions $\cosh\big((1-y)m_n\big)$ and $\sinh\big((1-y)m_n\big)/m_n$, where

$$m_n = \sqrt{n^2 + n^{2\alpha} - k^2}. \quad (5)$$

Because $m_n \to \infty$ as $n \to \infty$, these hyperbolic functions grow exponentially for $y < 1$. Therefore, high-frequency components in the final data are amplified severely when one reconstructs the solution away from $y = 1$. This is the fundamental reason for the ill-posedness of the backward problem.

In practical observations, measurement errors are unavoidable. In this paper, the final data are perturbed by Gaussian white noise:

$$g^\varepsilon(x) = g(x) + \varepsilon\xi(x), h^\varepsilon(x) = h(x) + \varepsilon\xi(x), \quad (6)$$

where $\varepsilon > 0$ is the noise intensity and $\xi$ is a Gaussian white noise process in $L^2(0,\pi)$. The white noise model is appropriate when random perturbations affect all Fourier modes and when the uncertainty is naturally described in a probabilistic Hilbert-space setting. Since white noise is not an $L^2$ function in the classical pointwise sense, the data must be interpreted through its Fourier coordinates. The error analysis is consequently formulated in expectation [9, 13].

The aim of this paper is to construct a stable approximation to the solution of (1)-(4) from the noisy data (6). We use the Fourier truncation method: only finitely many Fourier modes are retained, and the truncation level is selected as a function of the noise magnitude. The main contribution is a complete convergence analysis for the homogeneous fractional Helmholtz problem with Gaussian white noise final measurements. Although the source term is absent, the problem remains severely ill-posed because the instability is caused by the backward hyperbolic multipliers.

Compared with the nonlinear nonhomogeneous setting, the homogeneous case allows a more explicit treatment. There is no fixed-point equation and no nonlinear Lipschitz term. Nevertheless, the stochastic nature of the data and the exponential instability require careful balancing between the propagated noise error and the truncation bias. This balance leads to convergence rates of logarithmic or Holder type, depending on the smoothness condition imposed on the exact solution.

The paper is organized as follows. Section 2 recalls the functional setting, the spectral fractional Laplacian, Gaussian white noise, and the explicit solution representation. Section 3 proves the ill-posedness of the homogeneous backward problem. Section 4 introduces the Fourier truncation method and establishes basic stochastic estimates for the truncated noisy data. Section 5 derives convergence rates in the $L^2(0,\pi)$ norm. Section 6 proves an error estimate in the $H^q(0,\pi)$ norm. Section 7 gives concluding remarks.

**2. Preliminaries and exact solution**

We begin with the notation used throughout the paper. Let

$$L^2(0,\pi) = \left\{ v: (0,\pi) \to \mathbb{R}: \ v \text{ is measurable and } \int_0^\pi |v(x)|^2 \, dx < \infty \right\},$$

with inner product and norm

$$\langle v_1, v_2\rangle = \int_0^\pi v_1(x)v_2(x)\,dx, \|v\| = \left(\int_0^\pi |v(x)|^2\,dx\right)^{1/2}.$$

For $n \in \mathbb{N}$, set

$$\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx), x \in (0,\pi). \quad (7)$$

Then $\{\phi_n\}_{n\geq 1}$ is an orthonormal basis of $L^2(0,\pi)$.

**Definition 2.1 (Spectral fractional Laplacian).** For $v \in L^2(0,\pi)$ and $\alpha \in (0,1)$, the spectral fractional Laplacian $(-\Delta)^\alpha$ is defined by

$$(-\Delta)^\alpha v(x) = \sum_{n=1}^\infty n^{2\alpha}\langle v, \phi_n\rangle\phi_n(x), \quad (8)$$

whenever the series is meaningful in $L^2(0,\pi)$.

For $s > 0$, define

$$H^s(0,\pi) = \{v \in L^2(0,\pi): \sum_{n=1}^\infty n^{2s}|\langle v, \phi_n\rangle|^2 < \infty\}, \quad (9)$$

with norm

$$\|v\|_{H^s(0,\pi)} = (\sum_{n=1}^\infty n^{2s}|\langle v, \phi_n\rangle|^2)^{1/2}. \quad (10)$$

The space $H^s(0,\pi)$ is a Hilbert space with the corresponding inner product

$$\langle v_1, v_2\rangle_{H^s(0,\pi)} = \sum_{n=1}^\infty n^{2s}\langle v_1, \phi_n\rangle\langle v_2, \phi_n\rangle.$$

Let $\Omega$ be a probability space. The Bochner space $L^2\big(\Omega, L^2(0,\pi)\big)$ consists of all measurable mappings $v: \Omega \to L^2(0,\pi)$ such that

$$\mathbb{E}\|v\|^2 < \infty.$$

Its norm is

$$\|v\|_{L^2\left(\Omega, L^2(0,\pi)\right)} = \sqrt{\mathbb{E}\|v\|^2}.$$

We also use the space

$$V = \left\{v: [0,1] \to L^2\big(\Omega, L^2(0,\pi)\big): \sup_{0\leq y\leq 1}\sqrt{\mathbb{E}\|v(\cdot, y)\|^2} < \infty\right\},$$

with norm

$$\|v\|_V = \sup_{0\leq y\leq 1}\sqrt{\mathbb{E}\|v(\cdot, y)\|^2}.$$

**Definition 2.2 (Gaussian white noise).** A generalized random process $\xi$ is called a Gaussian white noise process on $L^2(0,\pi)$ if, for all $g_1, g_2 \in L^2(0,\pi)$, the random variables $\langle \xi, g_1 \rangle$ and $\langle \xi, g_2 \rangle$ are Gaussian, satisfy

$$\mathbb{E}\langle \xi, g_j \rangle = 0, \mathbb{E}\left|\langle \xi, g_j \rangle\right|^2 = \|g_j\|^2,$$

for $j = 1,2$, and

$$\mathrm{Cov}(\langle \xi, g_1 \rangle, \langle \xi, g_2 \rangle) = \langle g_1, g_2 \rangle.$$

**Lemma 2.3.** Let $\xi$ be a Gaussian white noise process in $L^2(0,\pi)$ and let $\{\phi_n\}_{n\geq 1}$ be defined by (7). If

$$\xi_n = \langle \xi, \phi_n \rangle, n \in \mathbb{N},$$

then $\{\xi_n\}_{n\geq 1}$ is a sequence of independent standard Gaussian random variables.

We shall use the elementary estimates

$$\cosh(zx) \leq e^{zx}, \frac{\sinh(zx)}{x} \leq e^{zx}, z \in [0,1],\ x > 0. \tag{11}$$

Since $\alpha \in (0,1)$ and $k \in \left(0, \sqrt{2}\right)$, the quantity $m_n$ in (5) is positive for every $n \geq 1$. Indeed,

$$m_1^2 = 1 + 1 - k^2 > 0.$$

Moreover,

$$m_n \leq \sqrt{2}\, n, n \geq 1. \tag{12}$$

We now derive the explicit representation of the solution.

**Theorem 2.4.** Assume that $g, h \in L^2(0,\pi)$. If the problem (1)-(4) has a solution $u \in C\left([0,1]; L^2(0,\pi)\right)$, then it is represented by

$$u(x,y) = \sum_{n=1}^{\infty} \left[ g_n \cosh\left((1-y)m_n\right) - h_n \frac{\sinh\left((1-y)m_n\right)}{m_n} \right] \phi_n(x), \tag{13}$$

where

$$g_n = \langle g, \phi_n \rangle, h_n = \langle h, \phi_n \rangle, m_n = \sqrt{n^2 + n^{2\alpha} - k^2}.$$

Conversely, whenever the series in (13) defines a function in $C\left([0,1]; L^2(0,\pi)\right)$ and the differentiated series is meaningful, the function given by (13) solves (1)-(4).

*Proof.* Write the solution in the sine basis as

$$u(x,y) = \sum_{n=1}^{\infty} u_n(y)\phi_n(x), u_n(y) = \langle u(\cdot, y), \phi_n \rangle.$$

Multiplying (1) by $\phi_n$ and integrating over $(0,\pi)$ give

$$u_n''(y) - (n^2 + n^{2\alpha} - k^2)u_n(y) = 0.$$

Thus

$$u_n''(y) - m_n^2 u_n(y) = 0.$$

The final conditions imply

$$u_n(1) = g_n, u_n'(1) = h_n.$$

Solving this second-order ordinary differential equation backward from $y = 1$ yields

$$u_n(y) = g_n \cosh\big((1-y)m_n\big) - h_n \frac{\sinh\big((1-y)m_n\big)}{m_n}.$$

Substitution into the Fourier expansion gives (13). The converse follows by direct substitution of the series into the equation and the boundary-final conditions. ▫

Formula (13) shows that the exact solution operator is diagonal in the Fourier basis. The multipliers are exponentially increasing when $y < 1$, and this growth is the source of the instability.

**3. Ill-posedness of the homogeneous problem**

In this section we show that the backward homogeneous fractional Helmholtz problem is not stable with respect to the final data. The proof is based on a sequence of random perturbations whose expected size tends to zero while the expected error of the corresponding solutions diverges.

Let $g, h \in L^2(0, \pi)$ and let $u$ be the exact solution corresponding to $(g, h)$. For each $p \in \mathbb{N}$, define perturbed data

$$g^p(x) = g(x) + \frac{1}{p^{3/4}} \xi_p \phi_p(x), h^p(x) = h(x), \quad (14)$$

where $\xi_p = \langle \xi, \phi_p \rangle$. By Lemma 2.3, $\mathbb{E}\xi_p^2 = 1$. Hence

$$\mathbb{E}\|g^p - g\|^2 = \frac{1}{p^{3/2}} \to 0, \mathbb{E}\|h^p - h\|^2 = 0. \quad (15)$$

Let $u^p$ denote the solution corresponding to $(g^p, h^p)$. By (13),

$$u^p(x,y) - u(x,y) = \frac{1}{p^{3/4}} \xi_p \cosh\Big((1-y)m_p\Big) \phi_p(x).$$

Therefore,

$$\mathbb{E}\|u^p(\cdot, y) - u(\cdot, y)\|^2 = \frac{\cosh^2\big((1-y)m_p\big)}{p^{3/2}}. \quad (16)$$

In particular, at $y = 0$,

$$\mathbb{E}\|u^p(\cdot, 0) - u(\cdot, 0)\|^2 = \frac{\cosh^2(m_p)}{p^{3/2}} \geq \frac{1}{4} \frac{e^{2m_p}}{p^{3/2}} \to \infty$$

as $p \to \infty$. Combining this fact with (15), we obtain the following result.

**Theorem 3.1.** The backward problem (1)-(4) is ill-posed in the sense of Hadamard. More precisely, there exists a sequence of final data $(g^p, h^p)$ converging to $(g, h)$ in mean square, while the corresponding solutions $u^p$ do not converge to $u$ in $C\big([0,1]; L^2(0,\pi)\big)$ in mean square. In fact,

$$\sup_{0\le y\le 1} \mathbb{E}\|u^p(\cdot, y) - u(\cdot, y)\|^2 \to \infty.$$

*Proof.* The assertion follows from (15) and (16). Since $m_p \to \infty$, the exponential factor $e^{2m_p}$ dominates the polynomial factor $p^{3/2}$. Thus the solution map from final data to the reconstructed state is not continuous. ▫

This theorem justifies the use of a regularization method. Without filtering high-frequency components, the reconstruction is dominated by random noise in the high Fourier modes.

**4. Fourier truncation regularization**

Let the noisy final data be given by (6). In Fourier coordinates,

$$(g^\varepsilon)_n = g_n + \varepsilon\xi_n, (h^\varepsilon)_n = h_n + \varepsilon\xi_n, \tag{17}$$

where $\xi_n = \langle \xi, \phi_n \rangle$. Since infinitely many white-noise coordinates cannot be used directly in $L^2(0,\pi)$, the first step is to truncate the noisy data. Let $N(\varepsilon)$ be a positive integer such that

$$\lim_{\varepsilon\to 0} N(\varepsilon) = \infty, \lim_{\varepsilon\to 0} \varepsilon^2 N(\varepsilon) = 0. \tag{18}$$

Define

$$g_{N(\varepsilon)}(x) = \sum_{n=1}^{N(\varepsilon)} (g^\varepsilon)_n \, \phi_n(x), h_{N(\varepsilon)}(x) = \sum_{n=1}^{N(\varepsilon)} (h^\varepsilon)_n \, \phi_n(x). \tag{19}$$

The following lemma estimates the mean square error of the truncated noisy data.

**Lemma 4.1.** Let $s > 0$ and let $g, h \in H^s(0,\pi)$. Assume that $N(\varepsilon)$ satisfies (18). Then

$$\mathbb{E}\|g_{N(\varepsilon)} - g\|^2 \le \varepsilon^2 N(\varepsilon) + \frac{1}{\big(N(\varepsilon)\big)^{2s}} \|g\|^2_{H^s(0,\pi)}, \tag{20}$$

$$\mathbb{E}\|h_{N(\varepsilon)} - h\|^2 \le \varepsilon^2 N(\varepsilon) + \frac{1}{\big(N(\varepsilon)\big)^{2s}} \|h\|^2_{H^s(0,\pi)}. \tag{21}$$

*Proof.* Using Parseval's identity and Lemma 2.3, we obtain

$$\mathbb{E}\|g_{N(\varepsilon)} - g\|^2 = \varepsilon^2 \mathbb{E} \sum_{n=1}^{N(\varepsilon)} \xi_n^2 + \sum_{n>N(\varepsilon)} |g_n|^2 = \varepsilon^2 N(\varepsilon) + \sum_{n>N(\varepsilon)} |g_n|^2.$$

Since $g \in H^s(0,\pi)$,

$$\sum_{n>N(\varepsilon)} |g_n|^2 \le \frac{1}{\big(N(\varepsilon)\big)^{2s}} \sum_{n>N(\varepsilon)} n^{2s}\,|g_n|^2 \le \frac{1}{\big(N(\varepsilon)\big)^{2s}} \|g\|^2_{H^s(0,\pi)}.$$

This proves (20). The proof of (21) is identical. ▫

Let $B(\varepsilon)$ be another positive integer satisfying $B(\varepsilon) \le N(\varepsilon)$ and $B(\varepsilon) \to \infty$ as $\varepsilon \to 0$. The Fourier truncation regularized solution is defined by

$$u^{\varepsilon}_{N(\varepsilon)}(x,y) = \sum_{n=1}^{B(\varepsilon)} \Big[ \big(g_{N(\varepsilon)}\big)_n \cosh\big((1-y)m_n\big) - \big(h_{N(\varepsilon)}\big)_n \frac{\sinh\big((1-y)m_n\big)}{m_n} \Big] \phi_n(x). \quad (22)$$

Because the sum in (22) is finite, $u^{\varepsilon}_{N(\varepsilon)}$ is well-defined as an element of $V$.

**Remark 4.2.** There are two truncation levels. The level $N(\varepsilon)$ is used to project the noisy data and to make the Gaussian white noise meaningful in $L^2(0,\pi)$. The level $B(\varepsilon)$ is used in the reconstruction formula to suppress the exponentially unstable high-frequency multipliers. In the convergence proofs below, we choose $N(\varepsilon)$ and $B(\varepsilon)$ so that the stochastic data error and the deterministic truncation bias are balanced.

For later use, we record the following elementary estimate.

**Lemma 4.3.** Let $s > 0$ and suppose that $\|g\|_{H^s(0,\pi)} \le M_1$, $\|h\|_{H^s(0,\pi)} \le M_1$. Then, for every $y \in [0,1]$,

$$\mathbb{E}\,\| \sum_{n=1}^{B(\varepsilon)} \Big[ \Big( \big(g_{N(\varepsilon)}\big)_n - g_n \Big) \cosh\big((1-y)m_n\big) - \Big( \big(h_{N(\varepsilon)}\big)_n - h_n \Big) \frac{\sinh\big((1-y)m_n\big)}{m_n} \Big] \phi_n \,\|^2 \quad (23)$$

$$\le 2e^{2(1-y)m_{B(\varepsilon)}} \left( 2\varepsilon^2 N(\varepsilon) + \frac{2M_1^2}{\big(N(\varepsilon)\big)^{2s}} \right). \quad (24)$$

*Proof.* Using $(a+b)^2 \le 2(a^2+b^2)$, Parseval's identity, and (11), we have

$$\mathbb{E} \sum_{n=1}^{B(\varepsilon)} \left[ \Big( \big(g_{N(\varepsilon)}\big)_n - g_n \Big) \cosh\big((1-y)m_n\big) - \Big( \big(h_{N(\varepsilon)}\big)_n - h_n \Big) \frac{\sinh\big((1-y)m_n\big)}{m_n} \right]^2$$

$$\le 2e^{2(1-y)m_{B(\varepsilon)}} \left( \mathbb{E} \sum_{n=1}^{B(\varepsilon)} \Big| \big(g_{N(\varepsilon)}\big)_n - g_n \Big|^2 + \mathbb{E} \sum_{n=1}^{B(\varepsilon)} \Big| \big(h_{N(\varepsilon)}\big)_n - h_n \Big|^2 \right)$$

$$\le 2e^{2(1-y)m_{B(\varepsilon)}} \big( \mathbb{E}\|g_{N(\varepsilon)} - g\|^2 + \mathbb{E}\|h_{N(\varepsilon)} - h\|^2 \big).$$

The result follows from Lemma 4.1 and the bounds on $g$ and $h$. ▫

## 5. Error estimates in the $L^2$ norm

We now establish convergence rates for the regularized solution. We impose Sobolev smoothness on the final data and additional source conditions on the exact solution. The

first condition leads to logarithmic-type convergence at the initial point and mixed rates for $0 < y \leq 1$. The second condition gives Holder-type convergence.

Throughout this section, let

$$N(\varepsilon) = \lfloor \varepsilon^{-\frac{2}{2s+1}} \rfloor, \tag{25}$$

for sufficiently small $\varepsilon \in (0,1)$, and choose

$$B(\varepsilon) = \lfloor \frac{s}{\sqrt{2}(2s+1)} \ln \frac{1}{\varepsilon} \rfloor. \tag{26}$$

For small $\varepsilon$, $B(\varepsilon) \leq N(\varepsilon)$, $B(\varepsilon) \to \infty$, and $N(\varepsilon) \to \infty$. Moreover, by (12),

$$e^{2(1-y)m_{B(\varepsilon)}} \leq e^{2\sqrt{2}(1-y)B(\varepsilon)} \leq \varepsilon^{-\frac{2s(1-y)}{2s+1}}. \tag{27}$$

**Theorem 5.1 (Logarithmic-type estimate).** Let $s > 0$ and suppose that $g, h \in H^s(0,\pi)$ satisfy

$$\|g\|_{H^s(0,\pi)} \leq M_1, \|h\|_{H^s(0,\pi)} \leq M_1.$$

Let $u$ be the exact solution of (1)-(4), and let $u^{\varepsilon}_{N(\varepsilon)}$ be defined by (22), with $N(\varepsilon)$ and $B(\varepsilon)$ chosen as in (25)-(26). Assume that there exist $q > 0$ and $Q_1 > 0$ such that

$$\sum_{n=1}^{\infty} n^{2q} e^{2yn} |u_n(y)|^2 \leq Q_1, \forall y \in [0,1], \tag{28}$$

where $u_n(y) = \langle u(\cdot, y), \phi_n \rangle$. Then, for every $y \in [0,1]$ and sufficiently small $\varepsilon$, there exists a constant $C_1 > 0$, independent of $\varepsilon$ and $y$, such that

$$\mathbb{E}\left\| u^{\varepsilon}_{N(\varepsilon)}(\cdot, y) - u(\cdot, y) \right\|^2 \leq C_1 \left( \varepsilon^{\frac{2s(1+y)}{2s+1}} + \left( \ln \frac{1}{\varepsilon} \right)^{-2q} \varepsilon^{\frac{\sqrt{2}sy}{2s+1}} \right). \tag{29}$$

In particular, at $y = 0$,

$$\mathbb{E}\left\| u^{\varepsilon}_{N(\varepsilon)}(\cdot, 0) - u(\cdot, 0) \right\|^2 \leq C_1 \left( \varepsilon^{\frac{2s}{2s+1}} + \left( \ln \frac{1}{\varepsilon} \right)^{-2q} \right). \tag{30}$$

*Proof.* We decompose the error into a propagated noise part and a truncation bias part. By Parseval's identity,

$$\mathbb{E}\left\| u^{\varepsilon}_{N(\varepsilon)}(\cdot, y) - u(\cdot, y) \right\|^2$$

$$\leq 2\mathbb{E} \sum_{n=1}^{B(\varepsilon)} \left[ \left( \left( g_{N(\varepsilon)} \right)_n - g_n \right) \cosh\left( (1-y)m_n \right) - \left( \left( h_{N(\varepsilon)} \right)_n - h_n \right) \frac{\sinh\left( (1-y)m_n \right)}{m_n} \right]^2$$

$$+2 \sum_{n > B(\varepsilon)} |u_n(y)|^2.$$

The first term is estimated by Lemma 4.3. Using (25), (27), and the elementary bounds

$$\varepsilon^2 N(\varepsilon) \leq C \varepsilon^{\frac{4s}{2s+1}}, \left( N(\varepsilon) \right)^{-2s} \leq C \varepsilon^{\frac{4s}{2s+1}},$$

we obtain

$$\mathbb{E}\sum_{n=1}^{B(\varepsilon)}\left[\left(\left(g_{N(\varepsilon)}\right)_n - g_n\right)\cosh\left((1-y)m_n\right) - \left(\left(h_{N(\varepsilon)}\right)_n - h_n\right)\frac{\sinh\left((1-y)m_n\right)}{m_n}\right]^2 \leq C\varepsilon^{-\frac{2s(1-y)}{2s+1}}\varepsilon^{\frac{4s}{2s+1}} = C\varepsilon^{\frac{2s(1+y)}{2s+1}}. \quad (31)$$

For the tail, condition (28) gives

$$\sum_{n>B(\varepsilon)}|u_n(y)|^2 = \sum_{n>B(\varepsilon)} n^{-2q}\, e^{-2yn}(n^{2q}e^{2yn}|u_n(y)|^2) \leq \left(B(\varepsilon)\right)^{-2q}e^{-2yB(\varepsilon)}Q_1. \quad (32)$$

By (26),

$$\left(B(\varepsilon)\right)^{-2q} \leq C\left(\ln\frac{1}{\varepsilon}\right)^{-2q}, e^{-2yB(\varepsilon)} \leq C\varepsilon^{\frac{\sqrt{2}sy}{2s+1}}.$$

Combining these estimates yields

$$\sum_{n>B(\varepsilon)}|u_n(y)|^2 \leq C\left(\ln\frac{1}{\varepsilon}\right)^{-2q}\varepsilon^{\frac{\sqrt{2}sy}{2s+1}}. \quad (33)$$

The desired estimate (29) follows from (31) and (33). Taking $y = 0$ gives (30). ▫

**Remark 5.2.** The estimate at $y = 0$ contains the logarithmic factor $\left(\ln(1/\varepsilon)\right)^{-2q}$. This is characteristic of severely ill-posed backward problems. The weaker condition (28) is not sufficient to compensate completely for the exponential amplification of high Fourier modes at the initial point.

We next impose a stronger source condition.

**Theorem 5.3 (Holder-type estimate).** Let $s > 0$ and suppose that $g, h \in H^s(0,\pi)$ satisfy

$$\|g\|_{H^s(0,\pi)} \leq M_1, \|h\|_{H^s(0,\pi)} \leq M_1.$$

Let $u$ be the exact solution of (1)-(4), and let $u^{\varepsilon}_{N(\varepsilon)}$ be defined by (22), with $N(\varepsilon)$ and $B(\varepsilon)$ chosen as in (25)-(26). Assume that there exist $r > 0$ and $Q_2 > 0$ such that

$$\sum_{n=1}^{\infty} e^{2rn}\,|u_n(y)|^2 \leq Q_2, \forall y \in [0,1]. \quad (34)$$

Then, for every $y \in [0,1]$ and sufficiently small $\varepsilon$, there exists a constant $C_2 > 0$, independent of $\varepsilon$ and $y$, such that

$$\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(\cdot,y) - u(\cdot,y)\right\|^2 \leq C_2\left(\varepsilon^{\frac{2s(1+y)}{2s+1}} + \varepsilon^{\frac{\sqrt{2}rs}{2s+1}}\right). \quad (35)$$

*Proof.* The propagated noise part is exactly the same as in the proof of Theorem 5.1, hence it is bounded by

$$C\varepsilon^{\frac{2s(1+y)}{2s+1}}.$$

It remains to estimate the deterministic truncation tail. Using (34), we obtain

$$\sum_{n>B(\varepsilon)} |u_n(y)|^2 = \sum_{n>B(\varepsilon)} e^{-2rn}\, e^{2rn}|u_n(y)|^2$$

$$\leq e^{-2rB(\varepsilon)} Q_2 \leq C\varepsilon^{\frac{\sqrt{2}rs}{2s+1}}. \tag{36}$$

Combining the two estimates gives (35). ▫

**Remark 5.4.** The convergence rate in Theorem 5.3 is of Holder type for all $y \in [0,1]$. The improvement over Theorem 5.1 comes from the stronger exponential source condition (34). This condition gives faster decay of the high-frequency Fourier coefficients of the exact solution.

## 6. Error estimate in the $H^q$ norm

We now derive an estimate in a stronger Sobolev norm. The argument follows the same decomposition as before, but the factor $n^{2q}$ must be controlled on the retained modes and on the tail.

**Theorem 6.1.** Let $q > 0$ and $s > 0$. Suppose that $g, h \in H^s(0,\pi)$ satisfy

$$\|g\|_{H^s(0,\pi)} \leq M_1, \|h\|_{H^s(0,\pi)} \leq M_1.$$

Let $u$ be the exact solution of (1)-(4), and let $u^{\varepsilon}_{N(\varepsilon)}$ be defined by (22), with $N(\varepsilon)$ and $B(\varepsilon)$ chosen as in (25)-(26). Assume that the exponential source condition (34) holds for some $r > 0$ and $Q_2 > 0$. Then, for every $y \in [0,1]$ and sufficiently small $\varepsilon$, there exists a constant $C_3 > 0$, independent of $\varepsilon$ and $y$, such that

$$\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(\cdot,y) - u(\cdot,y)\right\|^2_{H^q(0,\pi)} \leq C_3 \left(\ln\frac{1}{\varepsilon}\right)^{2q} \left(\varepsilon^{\frac{2s(1+y)}{2s+1}} + \varepsilon^{\frac{\sqrt{2}rs}{2s+1}}\right). \tag{37}$$

*Proof.* By the definition of the $H^q$ norm and the decomposition into retained and discarded modes,

$$\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(\cdot,y) - u(\cdot,y)\right\|^2_{H^q(0,\pi)}$$

$$= \mathbb{E}\sum_{n=1}^{B(\varepsilon)} n^{2q}\left[\left(\left(g_{N(\varepsilon)}\right)_n - g_n\right)\cosh\left((1-y)m_n\right) - \left(\left(h_{N(\varepsilon)}\right)_n - h_n\right)\frac{\sinh\left((1-y)m_n\right)}{m_n}\right]^2$$

$$+ \sum_{n>B(\varepsilon)} n^{2q}\,|u_n(y)|^2.$$

For the first term, since $n \leq B(\varepsilon)$ on the retained range,

$$\mathbb{E}\sum_{n=1}^{B(\varepsilon)} n^{2q}\left[\left(\left(g_{N(\varepsilon)}\right)_n - g_n\right)\cosh\left((1-y)m_n\right) - \left(\left(h_{N(\varepsilon)}\right)_n - h_n\right)\frac{\sinh\left((1-y)m_n\right)}{m_n}\right]^2$$

$$\leq \big(B(\varepsilon)\big)^{2q} C\varepsilon^{\frac{2s(1+y)}{2s+1}}. \tag{38}$$

For the tail, consider the function $G(\omega) = \omega^{2q}e^{-2r\omega}$ for $\omega > 0$. Since

$$G'(\omega) = \omega^{2q-1}e^{-2r\omega}(2q - 2r\omega),$$

$G$ is decreasing for $\omega > q/r$. As $B(\varepsilon) \to \infty$, for sufficiently small $\varepsilon$ we have $B(\varepsilon) > q/r$. Therefore, for all $n > B(\varepsilon)$,

$$n^{2q}e^{-2rn} \leq \big(B(\varepsilon)\big)^{2q} e^{-2rB(\varepsilon)}.$$

Using (34), we obtain

$$\sum_{n>B(\varepsilon)} n^{2q}\,|u_n(y)|^2 = \sum_{n>B(\varepsilon)} n^{2q}\, e^{-2rn}e^{2rn}|u_n(y)|^2$$

$$\leq \big(B(\varepsilon)\big)^{2q} e^{-2rB(\varepsilon)} Q_2 \leq C\big(B(\varepsilon)\big)^{2q} \varepsilon^{\frac{\sqrt{2}rs}{2s+1}}. \tag{39}$$

Since

$$B(\varepsilon) \leq C\ln\frac{1}{\varepsilon},$$

combining (38) and (39) yields (37). ▫

**Remark 6.2.** The factor $\big(\ln(1/\varepsilon)\big)^{2q}$ in (37) is natural. Reconstruction in $H^q(0,\pi)$ requires control of $q$ additional derivatives, and the retained Fourier modes contribute a factor of order $B(\varepsilon)^{2q}$. Since $B(\varepsilon)$ is logarithmic in $1/\varepsilon$, the additional loss is logarithmic.

**7. Discussion of the choice of parameters**

The choices (25) and (26) are made to balance two competing effects. The first effect is the stochastic data error. After projection onto the first $N(\varepsilon)$ modes, the expected squared error of each data set is bounded by

$$\varepsilon^2 N(\varepsilon) + N(\varepsilon)^{-2s}.$$

The value $N(\varepsilon) = \lfloor \varepsilon^{-2/(2s+1)} \rfloor$ balances these two terms and gives the rate

$$\varepsilon^{4s/(2s+1)}.$$

This is the standard bias-variance balance for estimating an $H^s$ function from white noise in a Fourier basis.

The second effect is the exponential instability of the backward solution operator. The largest retained multiplier is bounded by

$$e^{(1-y)m_{B(\varepsilon)}} \leq e^{\sqrt{2}(1-y)B(\varepsilon)}.$$

Thus $B(\varepsilon)$ cannot grow algebraically in $1/\varepsilon$; otherwise the noise would be amplified too strongly. A logarithmic truncation level is appropriate. The choice

$$B(\varepsilon) = \lfloor \frac{s}{\sqrt{2}(2s+1)} \ln \frac{1}{\varepsilon} \rfloor$$

ensures that the amplified noise remains controlled and leads to the factor

$$\varepsilon^{2s(1+y)/(2s+1)}.$$

When $y = 1$, the problem is stable because no backward continuation is needed. When $y = 0$, the reconstruction is most unstable. The rates in Theorems 5.1 and 5.3 reflect this transition from the final boundary to the initial side.

It is possible to modify the constants in $B(\varepsilon)$ to optimize a particular term in the error estimate. The present choice is simple, explicit, and sufficient to prove convergence. Other regularization methods, such as quasi-boundary value methods, Tikhonov regularization, or spectral filter functions, can also be applied. However, Fourier truncation has the advantage of being directly compatible with the spectral representation of the fractional Laplacian.

**8. Conclusion**

We have studied a backward final value problem for the homogeneous fractional Helmholtz equation with Gaussian white noise on the measured final data. The problem is severely ill-posed because the Fourier representation of the solution contains exponentially increasing hyperbolic multipliers. We proved this instability by constructing a sequence of random perturbations whose expected data error tends to zero while the corresponding reconstruction error diverges.

To stabilize the problem, we proposed a Fourier truncation method. The noisy data are first projected onto finitely many Fourier modes, and the reconstruction itself is further restricted to a logarithmic number of modes. This suppresses the high-frequency components that are responsible for the exponential instability. Under Sobolev smoothness assumptions on the exact data, the propagated stochastic error was estimated in expectation. Under an exponentially weighted source condition on the exact solution, we obtained logarithmic-type convergence. Under a stronger exponential source condition, we established a Holder-type convergence rate in the $L^2(0,\pi)$ norm. An additional estimate in the $H^q(0,\pi)$ norm was also derived, with a natural logarithmic loss caused by the stronger norm.

The homogeneous case considered here is important because it isolates the instability mechanism of the fractional Helmholtz backward operator without the additional complications caused by nonlinear source terms. The analysis also provides a useful reference point for more general nonhomogeneous and nonlinear models. Future work may include numerical experiments, adaptive parameter selection, multidimensional domains, and comparisons with other regularization strategies.

"**Acknowledgements**: This work was support by a grant of the Saigon University project CSA.2025.075".

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Ngày nhận bài: 16/9/2024 Ngày nhận đăng: 27/7/2026