

THE HIT PROBLEM FOR THE POLYNOMIAL ALGEBRA OF FIVE VARIABLES IN A FAMILY OF GENERIC DEGREE

Bài toán hit đối với đại số đa thức năm biến tại một họ bậc tổng quát

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ABSTRACT

Let E^k be an elementary abelian 2-group of rank k and let BE^k be the classifying space of E^k . Then, the cohomology algebra $P_k = H^*(BE^k; \mathbb{F}_2)$ is a polynomial algebra over the field \mathbb{F}_2 with two elements in k generators x_1, x_2, \dots, x_k , with the degree of each x_i being 1. Hence, this algebra is a module over the mod-2 Steenrod algebra, \mathcal{A} .

The *hit problem* of Frank Peterson asks for a minimal generating set for the polynomial algebra P_k as a module over the mod-2 Steenrod algebra \mathcal{A} . Equivalently, we want to find a vector space basis for $\mathbb{F}_2 \otimes_{\mathcal{A}} P_k$ in each degree. This is an open problem in Algebraic Topology.

In this paper, we explicitly determine a minimal set of \mathcal{A} -generators for P_5 in terms of the admissible monomials for the case of the generic degree $m = 2^{d+1} + 2^d - 2$ with $d \geq 6$.

Keywords: *Steenrod squares, Peterson hit problem, modular representation, polynomial algebra.*

TÓM TẮT

Ký hiệu E^k là một 2-nhóm aben sơ cấp hạng k và BE^k là không gian phân loại của E^k . Khi đó đại số đối đồng điều $P_k = H^*(BE^k; \mathbb{F}_2) = \mathbb{F}_2[x_1, x_2, \dots, x_k]$ là đại số đa thức k biến x_1, x_2, \dots, x_k , trên trường \mathbb{F}_2 có 2 phần tử, mỗi biến x_i có bậc 1. Đại số này được xét như một môđun trên đại số Steenrod mod-2, \mathcal{A} .

Bài toán hit của Peterson là bài toán về việc xác định một tập sinh cực tiểu cho đại số đa thức P_k được xét như một môđun trên đại số Steenrod mod-2. Một cách tương đương, đây là bài toán về việc tìm một cơ sở của không gian vectơ $\mathbb{F}_2 \otimes_{\mathcal{A}} P_k$ tại mỗi bậc. Đây là một bài toán mở trong chuyên ngành tô pô đại số được Frank Peterson đưa ra vào năm 1986.

Trong bài báo này, chúng tôi xác định tường minh một tập cực tiểu các phần tử sinh của \mathcal{A} -môđun P_5 theo các đơn thức được chấp nhận được tại dạng bậc tổng quát $m = 2^{d+1} + 2^d - 2$ với $d \geq 6$.

Từ khóa: *Bình phương Steenrod, bài toán hit của Peterson, biểu diễn modula, đại số đa thức.*

1. INTRODUCTION

Denote by $P_k := \mathbb{F}_2[x_1, x_2, \dots, x_k]$ the polynomial algebra over the field of two elements, \mathbb{F}_2 , in k generators x_1, x_2, \dots, x_k , each of degree 1. This algebra arises as the cohomology with coefficients in \mathbb{F}_2 of a classifying space of an elementary abelian 2-group of rank k .

Being the cohomology of a space, P_k is a module over the mod 2 Steenrod algebra \mathcal{A} . The action of \mathcal{A} on P_k can be determined by the elementary properties of the Steenrod squares Sq^i and subject to the Cartan formula (see Steenrod and Epstein [13]).

An element g in P_k is called *hit* if it belongs to \mathcal{A}^+P_k , where \mathcal{A}^+ is the augmentation ideal of \mathcal{A} . That means g can be written as a finite sum $g = \sum_{u>0} Sq^u(g_u)$ for suitable polynomials $g_u \in P_k$.

We study the *Peterson hit problem* of determining a minimal set of generators for the polynomial algebra P_k as a module over the Steenrod algebra. In other words, we want to determine a basis of the \mathbb{F}_2 -vector space

$$QP_k := P_k/\mathcal{A}^+P_k = \mathbb{F}_2 \otimes_{\mathcal{A}} P_k.$$

The hit problem was first studied by Peterson [6], Priddy [8], Singer [11] and Wood [23], who showed its relation to several classical problems in the Algebraic Topology. Then, this problem was studied by Carlisle-Wood [1], Crabb-Hubbuck [2], Kameko [3], Mothebe [4], Nam [5], Repka-Selick [9], Silverman [10], Singer [12], Walker and Wood [20], the present author [14, 15, 16] and others.

The hit problem was explicitly computed by Peterson [6] for $k = 1, 2$, by Kameko [3] for $k = 3$ and by us [14, 15] for $k = 4$. This problem is open for any $k > 4$. The results of the hit problem and its applications to representations of general linear groups have been presented in the books of Walker and Wood [21, 22].

For any nonnegative integer m , denote by $(P_k)_m$ and $(QP_k)_m$ the subspaces of degree m homogeneous polynomials in the spaces P_k and QP_k respectively. Set $\mu(m) = \min\{u \in \mathbb{Z} : \alpha(m + u) \leq u\}$, where $\alpha(a)$ denotes the number of one in dyadic expansion of a positive integer a . The following is an early result of the hit problem. Originally, it is Peterson's conjecture in [6].

Theorem 1.1 (See Wood [23]). *If $\mu(m) > k$, then $(QP_k)_m = 0$.*

An important tools in the study of the hit problem is Kameko's homomorphism $\widetilde{Sq}_*^0 : QP_k \rightarrow QP_k$. This homomorphism is induced by the \mathbb{F}_2 -linear map $\phi : P_k \rightarrow P_k$ given by $\phi(x) = u$, if $x = x_1x_2 \dots x_k u^2$ and $\phi(x) = 0$, otherwise, for any monomial $x \in P_k$. Note that ϕ is not an \mathcal{A} -homomorphism. However, $\phi Sq^{2s} = Sq^s \phi$, and $\phi Sq^{2s+1} = 0$ for any non-negative integer s .

Theorem 1.2 (See Kameko [3]). *Let q be a positive integer. If $\mu(2q + k) = k$, then*

$$(\widetilde{Sq}_*)_{(k,q)}^0 := \widetilde{Sq}_*^0 : (QP_k)_{2q+k} \longrightarrow (QP_k)_q$$

is an isomorphism of the \mathbb{F}_2 -vector spaces.

From Theorems 1.1 and 1.2, the hit problem is reduced to the case of degree m such that $\mu(m) < k$. For $\mu(m) = k - 1$, the problem was studied by Crabb-Hubbuck [2], Nam [5], Repka-Selick [9], Walker-Wood [20] and the present author [14, 15]. For $\mu(m) = k - 2$, it was studied in [16] which provides a new tool for studying the Peterson hit problem.

Recently, many authors study the hit problem for the case $k = 5$ and the problem has been explicitly determined for the case of degree m such that $\alpha(m + s) = 1$ with $s = \mu(m) < 5$ (see e.g. [18]).

In this paper, we study the hit problem for a case of $k = 5$ and the degree m such that $\alpha(m + s) = 2$. More precisely, we explicitly compute the space $(QP_k)_{2^{d+1}+2^d-2}$, with $d \geq 6$, in terms of the admissible monomials (see Section 2). We prove the following.

Theorem 1.3. *For any integer $d \geq 6$, there exist exactly 2480 admissible monomials of degree $2^{d+1} + 2^d - 2$ in P_5 . Consequently, $\dim(QP_5)_{(2^{d+1}+2^d-2)} = 2480$.*

The paper is organized as follows. In Section 2, we recall some needed definitions and results on the admissible monomials in P_k , the criteria of Singer and Silverman on the hit monomials. The detailed proof of Theorem 1.3 is presented in Section 3. Finally, in Section 4, we list the admissible monomials of the degree $2^{d+1} + 2^d - 2$ in P_5 which are used in the proof of the main result.

2. PRELIMINARIES ON THE HIT PROBLEM

In this section, we recall some needed definitions and results on the weight vector of a monomial and the admissible monomials from Kameko [3], Singer [12] and the present author [15, 16] which will be used in the proof of Theorem 1.3.

Definition 2.1. A weight vector ω is a sequence of non-negative integers $(\omega_1, \dots, \omega_i, \dots)$ such that $\omega_i = 0$ for $i \gg 0$.

Let $x = x_1^{t_1} x_2^{t_2} \dots x_k^{t_k} \in P_k$. We denote $\nu_j(x) = t_j, 1 \leq j \leq k$. We define two sequences associated with x by

$$\omega(x) = (\omega_1(x), \omega_2(x), \dots, \omega_i(x), \dots), \quad \sigma(x) = (\nu_1(x), \nu_2(x), \dots, \nu_k(x)),$$

where $\omega_i(x) = \sum_{1 \leq j \leq k} \alpha_{i-1}(\nu_j(x)), i \geq 1$. The sequences $\omega(x)$ and $\sigma(x)$ are called the weight vector and the exponent vector of x respectively.

The sets of weight vectors and exponent vectors respectively are given the left lexicographical order.

Definition 2.2. Let ω be a weight vector and g, h two polynomials of the same degree in P_k .

- i) $g \equiv h$ if and only if $g + h \in \mathcal{A}^+P_k$.
- ii) $g \equiv_\omega h$ if and only if $g + h \in \mathcal{A}^+P_k + P_k^-(\omega)$.

It is easy to see that, the relations \equiv and \equiv_ω are equivalence ones. Denote

$$QP_k(\omega) = P_k(\omega)/((\mathcal{A}^+P_k \cap P_k(\omega)) + P_k^-(\omega)).$$

Then, for any degree m , we have

$$(QP_k)_m \cong \bigoplus_{\deg \omega=m} QP_k(\omega). \tag{2.1}$$

Definition 2.3. Let u, v be monomials in P_k with $\deg u = \deg v$. We define $u < v$ if and only if one of the following conditions holds:

- i) $\omega(u) < \omega(v)$;
- ii) $\omega(u) = \omega(v)$ and $\sigma(u) < \sigma(v)$.

Definition 2.4. A monomial u in P_k is said to be inadmissible if there are monomials v_1, v_2, \dots, v_r such that $v_j < u$ for $j = 1, 2, \dots, r$ and $u + \sum_{j=1}^r v_j \in \mathcal{A}^+P_k$. A monomial u is said to be admissible if it is not inadmissible.

Obviously, the set of all the admissible monomials of degree m in P_k is a minimal set of \mathcal{A} -generators for P_k in degree m .

Denote by \mathcal{A}_s the sub-Hopf algebra of \mathcal{A} generated by Sq^i with $0 \leq i < 2^s$, and $\mathcal{A}_s^+ = \mathcal{A}^+ \cap \mathcal{A}_s$.

Definition 2.5. A monomial u in P_k is said to be strictly inadmissible if and only if there exist monomials v_1, v_2, \dots, v_r such that $v_j < u$, for $j = 1, 2, \dots, r$ and $u + \sum_{j=1}^r v_j \in \mathcal{A}_s^+P_k$ with $s = \max\{i : \omega_i(x) > 0\}$.

Theorem 2.6 (See Kameko [3], Sum [14]). *Let u, v, w be monomials in P_k such that $\omega_i(u) = 0$ for $i > r > 0$, $\omega_s(w) \neq 0$ and $\omega_i(w) = 0$ for $i > s > 0$.*

- i) *If w is inadmissible, then so is uw^{2^r} .*
- ii) *If w is strictly inadmissible, then so is wv^{2^s} .*

Proposition 2.7 (See [15, Proposition 4.3]). *Let x be an admissible monomial in P_k and let i_0 be a positive integer. Then we have*

- i) *If $\omega_{i_0}(x) = 0$, then $\omega_i(x) = 0$ for all $i > i_0$.*
- ii) *If $\omega_{i_0}(x) < k$, then $\omega_i(x) < k$ for all $i > i_0$.*

We recall a result of Singer [12] on the hit monomials in P_k .

Definition 2.8. A monomial z in P_k is called a spike if $\nu_j(z) = 2^{d_j} - 1$ for d_j a non-negative integer and $j = 1, 2, \dots, k$. If $d_1 > d_2 > \dots > d_{r-1} \geq d_r > 0$ and $d_j = 0$ for $j > r$, then z is called the minimal spike.

Clearly, all the spikes in P_k are admissible and their weight vectors are weakly decreasing. The following is a criterion for hit monomials in P_k .

Theorem 2.9 (See Singer [12, Theorem 1.2]). *Let $x \in P_k$ be a monomial of degree m with $\mu(m) \leq k$ and let z be the minimal spike of degree m . If $\omega(x) < \omega(z)$, then x is hit.*

Singer shows in [12] that this criterion is not enough to determine all hit monomials. So, we need Silverman's criterion for hit polynomials in P_k .

Theorem 2.10 (See Silverman [10, Theorem 1.2]). *Let p be a polynomial of the form fg^{2^d} for some homogeneous polynomials f and g . If $\deg f < (2^d - 1)\mu(\deg g)$, then p is hit.*

This result leads to a criterion in terms of the minimal spike which is a stronger version of Theorem 2.9.

Theorem 2.11 (See Walker and Wood [20, Theorem 14.1.3]). *Let $x \in P_k$ be a monomial of degree m such that $\mu(m) \leq k$ and let z be the minimal spike of degree m . If there is an index r such that $\sum_{i=1}^r 2^{i-1}\omega_i(x) < \sum_{i=1}^r 2^{i-1}\omega_i(z)$, then x is hit.*

Set $P_k^0 = \langle \{x_1^{t_1} \dots x_k^{t_k} \in P_k : t_1 \dots t_k = 0\} \rangle$, $P_k^+ = \langle \{x_1^{t_1} \dots x_k^{t_k} \in P_k : t_1 \dots t_k > 0\} \rangle$.

It is easy to see that P_k^0 and P_k^+ are the \mathcal{A} -submodules of P_k and $QP_k = QP_k^0 \oplus QP_k^+$. Here, $QP_k^0 = P_k^0/\mathcal{A}^+P_k^0$ and $QP_k^+ = P_k^+/\mathcal{A}^+P_k^+$.

We denote $\mathcal{N}_k = \{(i; I) : I = (i_1, i_2, \dots, i_r), 1 \leq i < i_1 < \dots < i_r \leq k, 0 \leq r < k\}$. For each $(i; I) \in \mathcal{N}_k$, we define the homomorphism of \mathcal{A} -algebras $p_{(i;I)} : P_k \rightarrow P_{k-1}$ by

$$p_{(i;I)}(x_j) = \begin{cases} x_j, & \text{if } 1 \leq j < i, \\ \sum_{s \in I} x_{s-1}, & \text{if } j = i, \\ x_{j-1}, & \text{if } i < j \leq k. \end{cases} \quad (2.2)$$

Lemma 2.12 (See [7]). *For any monomial u in P_k , we have $p_{(i;I)}(u) \in P_{k-1}(\omega(u))$.*

It is easy to see that if ω is a weight vector of degree m and $u \in P_k(\omega)$, then $p_{(i;I)}(u) \in P_{k-1}(\omega)$. Moreover, $p_{(i;I)}$ passes to homomorphisms

$$p_{(i;I)} : QP_k(\omega) \longrightarrow QP_{k-1}(\omega), \quad p_{(i;I)} : (QP_k)_m \longrightarrow (QP_{k-1})_m.$$

For $J = (j_1, j_2, \dots, j_s) : 1 \leq j_1 < \dots < j_s \leq k$, we define a monomorphism of \mathcal{A} -algebras $\theta_J : P_s \rightarrow P_k$ by substituting

$$\theta_J(x_t) = x_{j_t} \text{ for } 1 \leq t \leq s. \tag{2.3}$$

Obviously, for any weight vector ω of degree m , we have $Q\theta_J(P_s^+)(\omega) \cong QP_s^+(\omega)$ and $(Q\theta_J(P_s^+))_m \cong (QP_s^+)_m$ for $1 \leq s \leq k$, where $Q\theta_J(P_s^+) = \theta_J(P_s^+)/\mathcal{A}^+\theta_J(P_s^+)$. Hence, from a result in Wood [23] and (2.1), we get the following.

Proposition 2.13 (See Walker and Wood [21]). *Let ω be a weight vector of degree m . Then, we have*

$$\begin{aligned} \dim QP_k(\omega) &= \sum_{\mu(m) \leq s \leq k} \binom{k}{s} \dim QP_s^+(\omega), \\ \dim (QP_k)_m &= \sum_{\mu(m) \leq s \leq k} \binom{k}{s} \dim (QP_s^+)_m. \end{aligned}$$

Set $J_t = (1, \dots, \hat{t}, \dots, k)$ for $1 \leq t \leq k$.

Proposition 2.14 (See Mothebe and Uys [4]). *Let t, d be positive integers such that $1 \leq t \leq k$. If u is an admissible monomial in P_{k-1} , then so is $x_t^{2^d-1}\theta_{J_t}(u)$ in P_k .*

Notation 2.15. Denote by $B_k(m)$ the set of all admissible monomials of degree m in P_k . We set $B_k^0(m) = B_k(m) \cap P_k^0$, $B_k^+(m) = B_k(m) \cap P_k^+$.

For any weight vector ω of degree m , we denote

$$B_k(\omega) = B_k(m) \cap P_k(\omega), \quad B_k^+(\omega) = B_k^+(m) \cap P_k(\omega).$$

For a polynomial $g \in P_k$, denote by $[g]$ the class in QP_k represented by g . If ω is a weight vector and $g \in P_k(\omega)$, then we denote $[g]_\omega$ the class in $QP_k(\omega)$ represented by g . If ω is the minimal weight vector, then $[g]_\omega = [g]$.

For a subset T of P_k , we denote $[T] = \{[g] : g \in T\}$. For $T \subset P_k(\omega)$, we set $[T]_\omega = \{[g]_\omega : g \in T\}$. Then, $[B_k(\omega)]_\omega$ and $[B_k^+(\omega)]_\omega$, are the bases of the \mathbb{F}_2 -vector spaces $QP_k(\omega)$ and $QP_k^+(\omega) := QP_k(\omega) \cap QP_k^+$ respectively. Denote by $|T|$ the cardinal of T .

3. PROOF OF THEOREM 1.3

3.1. The weight vectors of the admissible monomial.

First of all, we determine the weight vectors of the admissible monomials of degree $m = 2^{d+1} + 2^d - 2$ in P_5 .

Lemma 3.1.1. *Let x be an admissible monomial of degree $2^{d+1} + 2^d - 2$ in P_5 with d a positive integer. If $d \geq 4$, then either $\omega(x) = (2)|^d|(1)$ or $\omega(x) = (4)|(3)|^{d-1}$ or $\omega(x) = (4)|(3)|^{d-2}|(1)|^2$.*

We need the following technical lemma.

Lemma 3.1.2. *Let (i, j, t, u, v) be an arbitrary permutation of $(1, 2, 3, 4, 5)$. The following monomials are strictly inadmissible:*

- i) $x_i^2 x_j^2 x_t^3 x_u^3; x_i^2 x_j x_t^2 x_u^2 x_v^3, i < j$.
- ii) $x_i^3 x_j^4 x_t^4 x_u^4 x_v^7; x_i^3 x_j^4 x_t^4 x_u^5 x_v^6; x_i x_j^2 x_t^6 x_u^6 x_v^7; x_i x_j^6 x_t^3 x_u^6 x_v^6, i < j < t$.
- iii) $x_i x_j^3 x_t^{14} x_u^{14} x_v^{14}; x_i x_j^{13} x_t^2 x_u^{14} x_v^{14}; x_i^3 x_j^5 x_t^{10} x_u^{14} x_v^{14}, x_i^3 x_j^{13} x_t^6 x_u^{10} x_v^{14}$.
- iv) $x_i^3 x_j^5 x_t^6 x_u^6 x_v^{26}$.

Proof. The proof of Part i) is easy. Part ii) is proved in [17]. For $x = x_i^3 x_j^5 x_t^6 x_u^6 x_v^{26}$, we have $\omega(x) = (2, 4, 3, 1, 1)$. By a direct computation, we get

$$\begin{aligned} x = & Sq^1 \left(x_i^3 x_j^5 x_t^6 x_u^6 x_v^{25} + x_i^3 x_j^5 x_t^6 x_u^8 x_v^{23} + x_i^3 x_j^5 x_t^8 x_u^6 x_v^{23} + x_i^3 x_j^5 x_t^{10} x_u^{12} x_v^{15} \right. \\ & + x_i^3 x_j^5 x_t^{12} x_u^{10} x_v^{15} + x_i^3 x_j^8 x_t^5 x_u^6 x_v^{23} + x_i^3 x_j^{10} x_t^5 x_u^{12} x_v^{15} + x_i^3 x_j^{10} x_t^{12} x_u^5 x_v^{15} \\ & + x_i^3 x_j^{12} x_t^5 x_u^{10} x_v^{15} + x_i^3 x_j^{12} x_t^{10} x_u^5 x_v^{15} + x_i^9 x_j^5 x_t^6 x_u^{10} x_v^{15} + x_i^9 x_j^5 x_t^{10} x_u^6 x_v^{15} \\ & + x_i^9 x_j^{10} x_t^5 x_u^6 x_v^{15} + x_i^{10} x_j^3 x_t^5 x_u^{12} x_v^{15} + x_i^{10} x_j^3 x_t^{12} x_u^5 x_v^{15} + x_i^{10} x_j^{12} x_t^3 x_u^5 x_v^{15} \\ & \left. + x_i^{12} x_j^3 x_t^5 x_u^{10} x_v^{15} + x_i^{12} x_j^3 x_t^{10} x_u^5 x_v^{15} + x_i^{12} x_j^{10} x_t^3 x_u^5 x_v^{15} \right) \\ & + Sq^2 \left(x_i^3 x_j^6 x_t^6 x_u^6 x_v^{23} + x_i^3 x_j^6 x_t^{10} x_u^{10} x_v^{15} + x_i^3 x_j^{10} x_t^6 x_u^{10} x_v^{15} + x_i^3 x_j^{10} x_t^{10} x_u^6 x_v^{15} \right. \\ & + x_1^{10} x_2^3 x_3^6 x_4^{10} x_5^{15} + x_1^{10} x_2^3 x_3^{10} x_4^6 x_5^{15} + x_1^{10} x_2^{10} x_3^3 x_4^6 x_5^{15} \left. \right) + Sq^4 \left(x_1^5 x_2^6 x_3^6 x_4^6 x_5^{19} \right) \\ & + Sq^8 \left(x_1^5 x_2^6 x_3^6 x_4^6 x_5^{15} \right) \pmod{P_5^-(2, 4, 3, 1, 1)} \end{aligned}$$

The above equality shows that the monomial x is strictly inadmissible. The proof of the monomials in Part iii) are carried out by a similar computation. \square

Proof of Lemma 3.1.1. Observe that $z = x_1^{2^{d+1}-1} x_2^{2^d-1}$ is the minimal spike of degree $2^{d+1} + 2^d - 2$ in P_5 and $\omega(z) = (2)|^d|(1)$. Since $2^{d+1} + 2^d - 2$ is even, using Theorem 2.9, we obtain either $\omega_1(x) = 2$ or $\omega_1(x) = 4$. If $\omega_1(x) = 4$, then $x = x_1 \dots \hat{x}_i \dots x_5 y^2$ with y an admissible monomial of degree $2^d + 2^{d-1} - 3$ in P_5 and $1 \leq i \leq 5$. By combining Proposition 2.7 and Lemma 3.1.1 in [16] we get either $\omega(y) = (3)|^{d-1}$ or $\omega(y) = (3)|^{d-2}|(1)|^2$, so $\omega(x) = (4)|(3)|^{d-1}$ or $\omega(x) = (4)|(3)|^{d-2}|(1)|^2$.

Suppose $\omega_1(x) = 2$. We prove $\omega(x) = (2)|^d|(1)$ by induction on $d \geq 4$. Since $\omega_1(x) = 2$, we have $x = x_i x_j y^2$ with $1 \leq i < j \leq 5$ and y an admissible monomial of degree $2^d + 2^{d-1} - 2$. For $d = 4$, by a direct computation we see that if $\omega(y) \neq (2)|^3|(1)$, then there is a monomial u as given in Lemma 3.1.2 such that $x = uz^{2^r}$ with $2 \leq r \leq 5$ and z a suitable monomial in P_5 . By Theorem 2.6, x is inadmissible. So, we get $\omega(y) =$

$(2)^3|(1)$ and $\omega(x) = (2)^4|(1)$. Suppose $d > 5$, by the inductive hypothesis, we have $\omega(y) = (2)^{d-1}|(1)$, hence $\omega(x) = (2)^d|(1)$. The lemma is proved. \square

From Lemma 3.1.1 we obtain the following.

Corollary 3.1.3. *For $m = 2^{d+1} + 2^d - 2$ with $d \geq 4$, we have*

$$(QP_5)_m \cong QP_5((2)^d|(1)) \oplus QP_5((4)|(3)^{d-1}) \oplus QP_5((4)|(3)^{d-2}|(1)^2).$$

The space $QP_5((4)|(3)^{d-1})$ has been determined in [18, Proposition 3.2.1].

Theorem 3.1.4. *For any $d \geq 6$, $B_5((4)|(3)^{d-1})$ is the set of 310 admissible monomials. Consequently, $\dim QP_5((4)|(3)^{d-1}) = 310$.*

The elements of the set $B_5((4)|(3)^{d-1})$ are explicitly presented in the Appendix of [18]. Thus, we need only to compute $QP_5((2)^d|(1))$ and $QP_5((4)|(3)^{d-2}|(1)^2)$.

3.2. Computation of $QP_5((2)^d|(1))$.

From Peterson [6], Kameko [3] and [15], we have

$$|B_2^+((2)^d|(1))| = 2, \quad |B_2^+((2)^d|(1))| = 8$$

and $|B_4^+((2)^d|(1))| = 26$. Hence, by Proposition 2.13 we obtain

$$|B_5^0((2)^d|(1))| = 2\binom{5}{2} + 8\binom{5}{3} + 26\binom{5}{2} = 230.$$

In this subsection, we prove the following.

Theorem 3.2.1. *For any $d \geq 4$, $B_5^+((2)^d|(1)) = \{a_{d,t} : 1 \leq t \leq 80\}$, where the monomials $a_t = a_{d,t}$ are listed in Section 4. Consequently,*

$$\dim QP_5^+((2)^d|(1)) = 80, \quad \dim QP_5((2)^d|(1)) = 310.$$

We need a technical lemma for the proof of the theorem.

Lemma 3.2.2. *Let (i, j, t, u, v) be an arbitrary permutation of $(1, 2, 3, 4, 5)$. The following monomials are strictly inadmissible:*

- i) $x_i^3 x_j^4 x_t^3, x_i^7 x_j^7 x_t^8, 1 \leq i < j < t \leq 5$.
- ii) $x_i x_j^6 x_t^8 x_u^7, x_i x_j^{10} x_t^4 x_u^7, i < j < t; x_i^3 x_j^5 x_t^8 x_u^6, x_i^7 x_j^9 x_t^2 x_u^4, i < j < t < u; x_i^3 x_j^3 x_t^4 x_u^{12}$.
- iii) $x_1^3 x_2^5 x_3^{10} x_4^{16} x_5^{12}, x_1^3 x_2^{13} x_3^2 x_4^{12} x_5^{16}, x_1^3 x_2^{13} x_3^2 x_4^{20} x_5^8, x_1^3 x_2^{13} x_3^{18} x_4^4 x_5^8, x_1^{15} x_2^{17} x_3^2 x_4^4 x_5^8$.

Proof. Parts i) and ii) follow from Kameko [3] and our work [15]. We prove Part iii) for $x = x_1^3 x_2^5 x_3^{10} x_4^{16} x_5^{12}$. By a direct computation using the Cartan formula we have

$$\begin{aligned} x &= x_1^2 x_2^3 x_3^9 x_4^{12} x_5^{20} + x_1^2 x_2^3 x_3^9 x_4^{20} x_5^{12} + x_1^2 x_2^5 x_3^9 x_4^{12} x_5^{18} + x_1^2 x_2^5 x_3^9 x_4^{18} x_5^{12} \\ &\quad + x_1^3 x_2^3 x_3^8 x_4^{12} x_5^{20} + x_1^3 x_2^3 x_3^8 x_4^{20} x_5^{12} + x_1^3 x_2^4 x_3^9 x_4^{12} x_5^{18} + x_1^3 x_2^4 x_3^9 x_4^{18} x_5^{12} \end{aligned}$$

$$\begin{aligned}
 &+ x_1^3 x_2^5 x_3^8 x_4^{12} x_5^{18} + x_1^3 x_2^5 x_3^8 x_4^{18} x_5^{12} + x_1^3 x_2^5 x_3^{10} x_4^{12} x_5^{16} + Sq^1 \left(x_1^3 x_2^3 x_3^5 x_4^{16} x_5^{18} \right. \\
 &+ x_1^3 x_2^3 x_3^5 x_4^{18} x_5^{16} + x_1^3 x_2^3 x_3^9 x_4^{12} x_5^{18} + x_1^3 x_2^3 x_3^9 x_4^{18} x_5^{12} \left. \right) + Sq^2 \left(x_1^2 x_2^3 x_3^9 x_4^{12} x_5^{18} \right. \\
 &+ x_1^2 x_2^3 x_3^9 x_4^{18} x_5^{12} + x_1^5 x_2^3 x_3^6 x_4^{12} x_5^{18} + x_1^5 x_2^3 x_3^6 x_4^{18} x_5^{12} \left. \right) + Sq^4 \left(x_1^3 x_2^3 x_3^6 x_4^{12} x_5^{18} \right. \\
 &+ x_1^3 x_2^3 x_3^6 x_4^{18} x_5^{12} + x_1^3 x_2^9 x_3^6 x_4^{12} x_5^{12} \left. \right) + Sq^8 \left(x_1^3 x_2^5 x_3^6 x_4^{12} x_5^{12} \right) \pmod{(P_5^-((2)|^d|(1)))}.
 \end{aligned}$$

Hence, the monomials x is strictly inadmissible. The other monomials are carried out by a similar computation. \square

Proof of Theorem 3.2.1. Let X be a monomial of wight vector $(2)|^d|(1)$ in P_5^+ with $d \geq 4$. Then we have $X = xx_j^{2^d}$ with x a monomial of weight vector $(2)|^d$ and $1 \leq j \leq 5$. From the monomials of $B_5((2)|^d)$ as listed in the Appendix of [18] we see that there exists uniquely a pair $J_x = (j_1^x, j_2^x)$ such that $1 \leq j_1^x < j_2^x \leq 5$ and $\nu_{j_i^x}(x) > 16$ for $i = 1, 2$. By a direct computation we see that if $j \neq j_1^x, j_2^x$, then there is a monomial w as given in Lemma 3.2.2 such that $x = yw^{2^{d+1-r}}$ with r an integer such that $2 \leq r \leq 5$, and y a suitable monomial of weight vector $(2)|^{d+1-r}$. By Theorem 2.6, x is inadmissible and we have a contradiction. Hence, either $j = j_1^x$ or $j = j_2^x$ and $x_j = \theta_{J_x}(x_u)$ for suitable $u = 1, 2$. Thus, we get

$$B_5((2)|^d|(1)) \subset A(d) := \{x\theta_{J_x}(x_u) : x \in B_5((2)|^d), u = 1, 2\}.$$

By a simple computation we see that $A(d) = B_5^0((2)|^d|(1)) \cup \{a_{d,t} : 1 \leq t \leq 80\}$, where the monomials $a_t = a_{d,t}$ are determined in Section 4.

We prove $B_5((2)|^d|(1)) = A(d)$ by proving that the set $\{[a_{d,t}] : 1 \leq t \leq 80\}$ is linearly independent in $QP_5((2)|^d|(1))$. Suppose there is a linear relation

$$\mathcal{T} := \sum_{1 \leq t \leq 80} \gamma_t a_{d,t} \equiv 0, \tag{3.1}$$

where $\gamma_t \in \mathbb{F}_2$. We denote $\gamma_{\mathbb{J}} = \sum_{t \in \mathbb{J}} \gamma_t$ for any $\mathbb{J} \subset \{t \in \mathbb{N} : 1 \leq t \leq 80\}$.

Let $v_{d,j}$, $1 \leq j \leq 26$, be as in Section 4 and the homomorphism $p_{(i;I)} : P_5 \rightarrow P_4$ defined by (2.2) for $k = 5$. By using Lemma 2.12, we see that $p_{(i;I)}$ passes to a homomorphism from $QP_5((2)|^d|(1))$ to $QP_4((2)|^d|(1))$. By applying $p_{(1;2)}$ and $p_{(4;5)}$ to (3.1), we obtain

$$\begin{aligned}
 p_{(1;2)}(\mathcal{T}) &\equiv \gamma_{\{13,16,21,23,27,30\}} v_1 + \gamma_{\{14,19,24,26,30\}} v_2 + \gamma_{15} v_3 + \gamma_{16} v_4 + \gamma_{17} v_5 \\
 &+ \gamma_{18} v_6 + \gamma_{19} v_7 + \gamma_{20} v_8 + \gamma_{21} v_9 + \gamma_{\{23,30\}} v_{10} + \gamma_{24} v_{11} + \gamma_{26} v_{12} + \gamma_{27} v_{13} \\
 &+ \gamma_{29} v_{14} + \gamma_{30} v_{15} + \gamma_{32} v_{16} + \gamma_{\{9,16,21,27\}} v_{17} + \gamma_{\{10,23\}} v_{18} + \gamma_{\{11,19,24,30\}} v_{19} \\
 &+ \gamma_{\{12,26\}} v_{20} + \gamma_{\{22,30\}} v_{21} + \gamma_{25} v_{22} + \gamma_{28} v_{23} + \gamma_{31} v_{24} + \gamma_{49} v_{25} + \gamma_{51} v_{26} \equiv 0, \\
 p_{(4;5)}(\mathcal{T}) &\equiv \gamma_7 v_1 + \gamma_8 v_2 + \gamma_{27} v_3 + \gamma_{28} v_4 + \gamma_{29} v_5 + \gamma_{30} v_6 + \gamma_{31} v_7 + \gamma_{32} v_8 + \gamma_{45} v_9 \\
 &+ \gamma_{46} v_{10} + \gamma_{47} v_{11} + \gamma_{48} v_{12} + \gamma_{49} v_{13} + \gamma_{50} v_{14} + \gamma_{51} v_{15} + \gamma_{52} v_{16} + \gamma_{65} v_{17} + \gamma_{66} v_{18} \\
 &+ \gamma_{67} v_{19} + \gamma_{68} v_{20} + \gamma_{80} v_{21} + \gamma_{74} v_{22} + \gamma_{75} v_{23} + \gamma_{76} v_{24} + \gamma_{77} v_{25} + \gamma_{78} v_{26} \equiv 0.
 \end{aligned}$$

From these equalities, we get

$$\gamma_t = 0 \text{ for } 7 \leq t \leq 32 \text{ or } 45 \leq t \leq 52 \text{ or } t \in \{65, 66, 67, 68, 74, 75, \dots, 78, 80\}. \quad (3.2)$$

Apply $p_{(3;4)}$ and $p_{(3;5)}$ to (3.1) and using (3.2) to obtain

$$\begin{aligned} p_{(3;4)}(\mathcal{T}) &\equiv \gamma_{\{1,70,73\}}v_1 + \gamma_{\{3,72,79\}}v_2 + \gamma_{35}v_9 + \gamma_{\{33,36,41,70,73\}}v_{10} + \gamma_{38}v_{11} + \gamma_{\{34,39,43,72,79\}}v_{12} \\ &\quad + \gamma_{55}v_{17} + \gamma_{\{53,56,61,70,73\}}v_{18} + \gamma_{58}v_{19} + \gamma_{\{54,59,63,72,79\}}v_{20} + \gamma_{69}v_{21} + \gamma_{71}v_{22} \equiv 0, \\ p_{(3;5)}(\mathcal{T}) &\equiv \gamma_{\{2,69,70\}}v_1 + \gamma_{\{4,71,72\}}v_2 + \gamma_{\{33,42,69\}}v_9 + \gamma_{37}v_{10} + \gamma_{\{34,44,71\}}v_{11} + \gamma_{40}v_{12} \\ &\quad + \gamma_{\{53,62,69\}}v_{17} + \gamma_{57}v_{18} + \gamma_{\{54,64,71\}}v_{19} + \gamma_{60}v_{20} + \gamma_{70}v_{21} + \gamma_{72}v_{22} \equiv 0. \end{aligned}$$

Computing from the above relations gives

$$\begin{aligned} \gamma_t &= 0 \text{ for } t \in \{2, 4, 35, 37, 38, 40, 55, 57, 58, 60, 69, 70, 71, 72\}, \\ \gamma_t &= \gamma_u \text{ for } (t, u) \in \{(33, 42), (34, 44), (53, 62), (1, 73), (3, 79)\}. \end{aligned} \quad (3.3)$$

By a direct computation using (3.2) and (3.3) we obtain

$$\begin{aligned} p_{(1;4)}(\mathcal{T}) &\equiv \gamma_{\{1,5,33,61\}}v_1 + \gamma_{\{3,6,34,63\}}v_2 + \gamma_{\{5,36,61\}}v_4 + \gamma_{\{6,39,63\}}v_7 \\ &\quad + \gamma_{\{54,64\}}v_8 + \gamma_{\{1,5,33,61\}}v_{10} + \gamma_{\{3,6,34,63\}}v_{12} + \gamma_{41}v_{21} + \gamma_{43}v_{22} \equiv 0, \\ p_{(2;3)}(\mathcal{T}) &\equiv \gamma_{\{1,5,36,53\}}v_1 + \gamma_{\{3,6,39,54\}}v_2 + \gamma_{\{5,36,61\}}v_9 + \gamma_{\{1,5,36,53\}}v_{10} \\ &\quad + \gamma_{\{6,39,63\}}v_{11} + \gamma_{\{3,6,39,64\}}v_{12} + \gamma_{56}v_{21} + \gamma_{59}v_{22} \equiv 0, \\ p_{(2;4)}(\mathcal{T}) &\equiv \gamma_{\{5,41,53\}}v_1 + \gamma_{\{6,43,64\}}v_2 + \gamma_{\{1,5,41,56\}}v_4 + \gamma_{\{3,6,43,59\}}v_7 \\ &\quad + \gamma_{\{5,41,53\}}v_{10} + \gamma_{\{6,43,64\}}v_{12} + \gamma_{\{54,64\}}v_{20} + \gamma_{61}v_{21} + \gamma_{63}v_{22} \equiv 0. \end{aligned}$$

From the above relations one gets

$$\begin{aligned} \gamma_t &= 0 \text{ for } t \in \{33, 34, 41, 42, 43, 44, 56, 59, 61, 63\}, \\ \gamma_t &= \gamma_u \text{ for } (t, u) \in \{(1, 5), (3, 6), (1, 36), (3, 39), (1, 53), (3, 54), (3, 64)\}. \end{aligned} \quad (3.4)$$

By applying $p_{(2;5)}$ to (3.1) and using (3.2), (3.3) and (3.4), we obtain

$$p_{(2;5)}(\mathcal{T}) \equiv \gamma_1v_4 + \gamma_3v_7 + \gamma_1v_{21} + \gamma_3v_{22} \equiv 0. \quad (3.5)$$

This equality implies $\gamma_1 = \gamma_3 = 0$. Hence, we get $\gamma_t = 0$ for all t , $1 \leq t \leq 80$. The proof is completed. \square

3.3. Computation of $QP_5^+((4)|(3)|^{d-2}|(1)|^2)$.

For $d \geq 6$, by using Proposition 3.2.1 in [18] we see that for any $x \in B_5((4)|(3)|^{d-2})$, there exists uniquely a sequence $\bar{J}_x = (j_1^x, j_2^x, j_3^x)$ such that $1 \leq j_1^x < j_2^x < j_3^x \leq 5$ and $\nu_{j_i^x}(x) > 16$ for $t = 1, 2, 3$. Consider the map $\theta_J : P_r \rightarrow P_5$ defined by (2.3) for $k = 5$ and $r < 5$. The main result of this subsection is the following.

Theorem 3.3.1. *For any $d \geq 6$, we have*

$$B_5((4)|(3)|^{d-2}|(1)|^2) = \left\{ x\theta_{\bar{J}_x} \left(y^{2^{d-1}} \right) : x \in B_5((4)|(3)|^{d-2}), y \in B_3((1)|^2) \right\}.$$

By Kameko [3], we have $|B_3((1)|^2)| = 6$. Hence, from the above proposition we obtain $\dim QP_5((4)|(3)|^{d-2}|(1)|^2) = 1860$. By combining Propositions 3.1.4, 3.2.1 and 3.3.1 we get $\dim(QP_5)_{(2^{d+1}+2^{d-2})} = 310 + 310 + 1860 = 2480$ for any $d \geq 6$. Thus, Theorem 1.3 is proved.

In the remaining part of the section, we prove Theorem 3.3.1.

Proof of Theorem 3.3.1. Let $d \geq 6$ and let $\bar{x} \in P_5((4)|(3)|^{d-2}|(1)|^2)$ be an admissible monomial. Then \bar{x} is of the form $\bar{x} = x.p^{2^{d-1}}$ with x a monomial of weight vector $(4)|(3)|^{d-2}$ and p a monomial of weight vector $(1)|^2$. From the proof of Proposition 3.1.4 in [18] we see that if $x \notin B_5((4)|(3)|^{d-2})$, then x is strictly inadmissible, hence by Theorem 2.6, \bar{x} is also strictly inadmissible. This implies a contradiction and the monomials x and p are admissible.

By a direct computation we see that if there is $1 \leq j \leq 5$ such that $j \neq j_1^x, j_2^x, j_3^x$ and $\nu_j(p) > 0$, then there is a monomial w as given in one of lemmas from Lemma 3.2.1 to Lemma 3.2.4 in our work [19] such that $\bar{x} = h.w^{2^{d-u}}$ with u a positive integer, $2 \leq u \leq 6$, and h a monomial of weight vector $(4)|(3)|^{d-u}$. By Theorem 2.6, \bar{x} is inadmissible and we have a contradiction. Hence $\nu_j(p) = 0$ for $j \neq j_1^x, j_2^x, j_3^x$ and $p = \theta_{J_x}(y)$ with suitable monomial y in P_3 . Thus, we obtain

$$B_5((4)|(3)|^{d-2}|(1)|^2) \subset \left\{ x\theta_{\bar{J}_x}(y)^{2^{d-1}} : x \in B_5((4)|(3)|^{d-2}), y \in B_3((1)|^2) \right\}.$$

Now we prove that the set in the right hand side of the last relation is a minimal set of \mathcal{A} -generators for $P_5(\bar{\omega}_d)$ with $\bar{\omega}_d = (4)|(3)|^{d-2}|(1)|^2$. Suppose there is a linear relation

$$S := \sum_{x \in B_5^+((4)|(3)|^{d-2}); y \in B_3((1)|^2)} \gamma_{x,y} \left(x\theta_{J_x}(y)^{2^{d-1}} \right) \equiv_{\bar{\omega}_d} 0,$$

where $\gamma_{x,y} \in \mathbb{F}_2$. Let $B_4(\bar{\omega}_d)$ be as in our work [15] and the homomorphism $p_{(i;I)} : P_5 \rightarrow P_4$ which is defined by (2.2) for $k = 5$. Note that $|B_4(\bar{\omega}_d)| = 90$. From Lemma 2.12, we see that $p_{(i;I)}$ passes to a homomorphism from $QP_5(\bar{\omega}_d)$ to $QP_4(\bar{\omega}_d)$. By a direct computation similar to the one in the proof of Proposition 3.2.1, we compute $p_{(i;I)}(S)$ in terms of the monomials in $B_4(\bar{\omega}_d) \pmod{(\mathcal{A}^+P_4 + P_4^-(\bar{\omega}_d))}$. By computing from the relations $p_{(i;I)}(S) \equiv_{\bar{\omega}_d} 0$ with all $(i;I) \in \mathcal{N}_5$ and $I \neq (2, 3, 4, 5)$, we obtain $\gamma_{x,y} = 0$ for all $x \in B_5((4)|(3)|^{d-2})$ and $y \in B_3((1)|^2)$. The proof is completed. \square

4. THE LIST OF THE ADMISSIBLE MONOMIALS

The set $B_5((4)|(3)^{d-1})$ is explicitly determined in the Appendix of the work [18] and the set $B_5((4)|(3)^{d-2}(1)^2)$ is determined in Theorem 3.3.1. So, in this section, we list all elements of $B_5((2)^d|(1))$.

From Peterson [6] we have

$$B_2((2)^d|(1)) = B_2^+((2)^d|(1)) = \{x_1^{2^d-1}x_2^{2^{d+1}-1}, x_1^{2^{d+1}-1}x_2^{2^d-1}\}.$$

From Kameko [3] we have $B_3^+((2)^d|(1)) = \{u_j = u_{d,j} : 1 \leq j \leq 8\}$, where

$$\begin{aligned} u_1 &= x_1x_2^{2^d-2}x_3^{2^{d+1}-1} & u_2 &= x_1x_2^{2^d-1}x_3^{2^{d+1}-2} & u_3 &= x_1x_2^{2^{d+1}-2}x_3^{2^d-1} & u_4 &= x_1x_2^{2^{d+1}-1}x_3^{2^d-2} \\ u_5 &= x_1^3x_2^{2^d-3}x_3^{2^{d+1}-2} & u_6 &= x_1^3x_2^{2^{d+1}-3}x_3^{2^d-2} & u_7 &= x_1^{2^d-1}x_2x_3^{2^{d+1}-2} & u_8 &= x_1^{2^{d+1}-1}x_2x_3^{2^d-2} \end{aligned}$$

From our work [15] we have $B_4^+((2)^d|(1)) = \{v_j = v_{d,j} : 1 \leq j \leq 26\}$, where

$$\begin{aligned} v_1 &= x_1x_2x_3^{2^d-2}x_4^{2^{d+1}-2} & v_2 &= x_1x_2x_3^{2^{d+1}-2}x_4^{2^d-2} & v_3 &= x_1x_2^2x_3^{2^d-4}x_4^{2^{d+1}-1} \\ v_4 &= x_1x_2^2x_3^{2^d-3}x_4^{2^{d+1}-2} & v_5 &= x_1x_2^2x_3^{2^d-1}x_4^{2^{d+1}-4} & v_6 &= x_1x_2^2x_3^{2^{d+1}-4}x_4^{2^d-1} \\ v_7 &= x_1x_2^2x_3^{2^{d+1}-3}x_4^{2^d-2} & v_8 &= x_1x_2^2x_3^{2^{d+1}-1}x_4^{2^d-4} & v_9 &= x_1x_2^3x_3^{2^d-4}x_4^{2^{d+1}-2} \\ v_{10} &= x_1x_2^3x_3^{2^d-2}x_4^{2^{d+1}-4} & v_{11} &= x_1x_2^3x_3^{2^{d+1}-4}x_4^{2^d-2} & v_{12} &= x_1x_2^3x_3^{2^{d+1}-2}x_4^{2^d-4} \\ v_{13} &= x_1x_2^{2^d-2}x_3x_4^{2^{d+1}-2} & v_{14} &= x_1x_2^{2^d-1}x_3x_4^{2^{d+1}-4} & v_{15} &= x_1x_2^{2^{d+1}-2}x_3x_4^{2^d-2} \\ v_{16} &= x_1x_2^{2^{d+1}-1}x_3^2x_4^{2^d-4} & v_{17} &= x_1^3x_2x_3^{2^d-4}x_4^{2^{d+1}-2} & v_{18} &= x_1^3x_2x_3^{2^d-2}x_4^{2^{d+1}-4} \\ v_{19} &= x_1^3x_2x_3^{2^{d+1}-4}x_4^{2^d-2} & v_{20} &= x_1^3x_2x_3^{2^{d+1}-2}x_4^{2^d-4} & v_{21} &= x_1^3x_2^5x_3^{2^d-6}x_4^{2^{d+1}-4} \\ v_{22} &= x_1^3x_2^5x_3^{2^{d+1}-6}x_4^{2^d-4} & v_{23} &= x_1^3x_2^{2^d-3}x_3^2x_4^{2^{d+1}-4} & v_{24} &= x_1^3x_2^{2^{d+1}-3}x_3^2x_4^{2^d-4} \\ v_{25} &= x_1^{2^d-1}x_2x_3^2x_4^{2^{d+1}-4} & v_{26} &= x_1^{2^{d+1}-1}x_2x_3^2x_4^{2^d-4} \end{aligned}$$

According to Proposition 2.13, we have $|B_5^0((2)^d|(1))| = 2\binom{5}{2} + 8\binom{5}{3} + 26\binom{5}{4} = 230$, and $B_5^0((2)^d|(1)) = \bigcup_{\ell(J)=s; 2 \leq s \leq 4} \theta_J(B_s^+((2)^d|(1)))$.

$B_5((2)^d|(1)) = B_5^0((2)^d|(1)) \cup \{a_{d,j} : 1 \leq j \leq 80\}$, where the monomials $a_j = a_{d,j}$ are determined as follows. For $d \geq 4$,

$$\begin{aligned} a_1 &= x_1x_2x_3^2x_4^{2^d-4}x_5^{2^{d+1}-2} & a_2 &= x_1x_2x_3^2x_4^{2^d-2}x_5^{2^{d+1}-4} & a_3 &= x_1x_2x_3^2x_4^{2^{d+1}-4}x_5^{2^d-2} \\ a_4 &= x_1x_2x_3^2x_4^{2^{d+1}-2}x_5^{2^d-4} & a_5 &= x_1x_2x_3^6x_4^{2^d-6}x_5^{2^{d+1}-4} & a_6 &= x_1x_2x_3^6x_4^{2^{d+1}-6}x_5^{2^d-4} \\ a_7 &= x_1x_2x_3^2x_4^{2^d-2}x_5^{2^{d+1}-4} & a_8 &= x_1x_2x_3^2x_4^{2^{d+1}-2}x_5^{2^d-4} & a_9 &= x_1x_2^2x_3x_4^{2^d-4}x_5^{2^{d+1}-2} \\ a_{10} &= x_1x_2^2x_3x_4^{2^d-2}x_5^{2^{d+1}-4} & a_{11} &= x_1x_2^2x_3x_4^{2^{d+1}-4}x_5^{2^d-2} & a_{12} &= x_1x_2^2x_3x_4^{2^{d+1}-2}x_5^{2^d-4} \\ a_{13} &= x_1x_2^2x_3^3x_4^{2^d-4}x_5^{2^{d+1}-4} & a_{14} &= x_1x_2^2x_3^3x_4^{2^{d+1}-4}x_5^{2^d-4} & a_{15} &= x_1x_2^2x_3^4x_4^{2^d-8}x_5^{2^{d+1}-1} \\ a_{16} &= x_1x_2^2x_3^4x_4^{2^d-7}x_5^{2^{d+1}-2} & a_{17} &= x_1x_2^2x_3^4x_4^{2^d-1}x_5^{2^{d+1}-8} & a_{18} &= x_1x_2^2x_3^4x_4^{2^{d+1}-8}x_5^{2^d-1} \\ a_{19} &= x_1x_2^2x_3^4x_4^{2^{d+1}-7}x_5^{2^d-2} & a_{20} &= x_1x_2^2x_3^5x_4^{2^{d+1}-1}x_5^{2^d-8} & a_{21} &= x_1x_2^2x_3^5x_4^{2^d-8}x_5^{2^{d+1}-2} \\ a_{22} &= x_1x_2^2x_3^5x_4^{2^d-6}x_5^{2^{d+1}-4} & a_{23} &= x_1x_2^2x_3^5x_4^{2^d-2}x_5^{2^{d+1}-8} & a_{24} &= x_1x_2^2x_3^5x_4^{2^{d+1}-8}x_5^{2^d-2} \\ a_{25} &= x_1x_2^2x_3^5x_4^{2^{d+1}-6}x_5^{2^d-4} & a_{26} &= x_1x_2^2x_3^5x_4^{2^{d+1}-2}x_5^{2^d-8} & a_{27} &= x_1x_2^2x_3^2x_4^{2^d-4}x_5^{2^{d+1}-2} \\ a_{28} &= x_1x_2^2x_3^2x_4^{2^d-3}x_5^{2^{d+1}-4} & a_{29} &= x_1x_2^2x_3^2x_4^{2^d-1}x_5^{2^{d+1}-8} & a_{30} &= x_1x_2^2x_3^2x_4^{2^{d+1}-4}x_5^{2^d-2} \\ a_{31} &= x_1x_2^2x_3^2x_4^{2^{d+1}-3}x_5^{2^d-4} & a_{32} &= x_1x_2^2x_3^2x_4^{2^{d+1}-1}x_5^{2^d-8} & a_{33} &= x_1x_2^3x_3^2x_4^{2^d-4}x_5^{2^{d+1}-4} \end{aligned}$$

$$\begin{aligned}
 a_{34} &= x_1 x_2^3 x_3^2 x_4^{2^{d+1}-4} x_5^{2^d-4} & a_{35} &= x_1 x_2^3 x_3^4 x_4^{2^d-8} x_5^{2^{d+1}-2} & a_{36} &= x_1 x_2^3 x_3^4 x_4^{2^d-6} x_5^{2^{d+1}-4} \\
 a_{37} &= x_1 x_2^3 x_3^4 x_4^{2^d-2} x_5^{2^{d+1}-8} & a_{38} &= x_1 x_2^3 x_3^4 x_4^{2^{d+1}-8} x_5^{2^d-2} & a_{39} &= x_1 x_2^3 x_3^4 x_4^{2^{d+1}-6} x_5^{2^d-4} \\
 a_{40} &= x_1 x_2^3 x_3^4 x_4^{2^{d+1}-2} x_5^{2^d-8} & a_{41} &= x_1 x_2^3 x_3^6 x_4^{2^d-8} x_5^{2^{d+1}-4} & a_{42} &= x_1 x_2^3 x_3^6 x_4^{2^d-4} x_5^{2^{d+1}-8} \\
 a_{43} &= x_1 x_2^3 x_3^6 x_4^{2^{d+1}-8} x_5^{2^d-4} & a_{44} &= x_1 x_2^3 x_3^6 x_4^{2^{d+1}-4} x_5^{2^d-8} & a_{45} &= x_1 x_2^3 x_3^{2^d-4} x_4^2 x_5^{2^{d+1}-4} \\
 a_{46} &= x_1 x_2^3 x_3^{2^d-2} x_4^4 x_5^{2^{d+1}-8} & a_{47} &= x_1 x_2^3 x_3^{2^{d+1}-4} x_4^2 x_5^{2^d-4} & a_{48} &= x_1 x_2^3 x_3^{2^{d+1}-2} x_4^4 x_5^{2^d-8} \\
 a_{49} &= x_1 x_2^{2^d-2} x_3^2 x_4^2 x_5^{2^{d+1}-4} & a_{50} &= x_1 x_2^{2^d-1} x_3^2 x_4^4 x_5^{2^{d+1}-8} & a_{51} &= x_1 x_2^{2^{d+1}-2} x_3 x_4^2 x_5^{2^d-4} \\
 a_{52} &= x_1 x_2^{2^{d+1}-1} x_3^2 x_4^4 x_5^{2^d-8} & a_{53} &= x_1^3 x_2 x_3^2 x_4^{2^d-4} x_5^{2^{d+1}-4} & a_{54} &= x_1^3 x_2 x_3^2 x_4^{2^{d+1}-4} x_5^{2^d-4} \\
 a_{55} &= x_1^3 x_2 x_3^4 x_4^{2^d-8} x_5^{2^{d+1}-2} & a_{56} &= x_1^3 x_2 x_3^4 x_4^{2^d-6} x_5^{2^{d+1}-4} & a_{57} &= x_1^3 x_2 x_3^4 x_4^{2^d-2} x_5^{2^{d+1}-8} \\
 a_{58} &= x_1^3 x_2 x_3^4 x_4^{2^{d+1}-8} x_5^{2^d-2} & a_{59} &= x_1^3 x_2 x_3^4 x_4^{2^{d+1}-6} x_5^{2^d-4} & a_{60} &= x_1^3 x_2 x_3^4 x_4^{2^{d+1}-2} x_5^{2^d-8} \\
 a_{61} &= x_1^3 x_2 x_3^6 x_4^{2^d-8} x_5^{2^{d+1}-4} & a_{62} &= x_1^3 x_2 x_3^6 x_4^{2^d-4} x_5^{2^{d+1}-8} & a_{63} &= x_1^3 x_2 x_3^6 x_4^{2^{d+1}-8} x_5^{2^d-4} \\
 a_{64} &= x_1^3 x_2 x_3^6 x_4^{2^{d+1}-4} x_5^{2^d-8} & a_{65} &= x_1^3 x_2 x_3^{2^d-4} x_4^2 x_5^{2^{d+1}-4} & a_{66} &= x_1^3 x_2 x_3^{2^d-2} x_4^4 x_5^{2^{d+1}-8} \\
 a_{67} &= x_1^3 x_2 x_3^{2^{d+1}-4} x_4^2 x_5^{2^d-4} & a_{68} &= x_1^3 x_2 x_3^{2^{d+1}-2} x_4^4 x_5^{2^d-8} & a_{69} &= x_1^3 x_2^5 x_3^2 x_4^{2^d-8} x_5^{2^{d+1}-4} \\
 a_{70} &= x_1^3 x_2^5 x_3^2 x_4^{2^d-4} x_5^{2^{d+1}-8} & a_{71} &= x_1^3 x_2^5 x_3^2 x_4^{2^{d+1}-8} x_5^{2^d-4} & a_{72} &= x_1^3 x_2^5 x_3^2 x_4^{2^{d+1}-4} x_5^{2^d-8} \\
 a_{73} &= x_1^3 x_2^5 x_3^{10} x_4^{2^d-12} x_5^{2^{d+1}-8} & a_{74} &= x_1^3 x_2^5 x_3^{2^{d+1}-6} x_4^4 x_5^{2^d-8} & a_{75} &= x_1^3 x_2^{2^d-3} x_3^2 x_4^4 x_5^{2^{d+1}-8} \\
 a_{76} &= x_1^3 x_2^{2^{d+1}-3} x_3^2 x_4^4 x_5^{2^d-8} & a_{77} &= x_1^{2^d-1} x_2 x_3^2 x_4^4 x_5^{2^{d+1}-8} & a_{78} &= x_1^{2^{d+1}-1} x_2 x_3^2 x_4^4 x_5^{2^d-8}
 \end{aligned}$$

For $d = 4$, $a_{79} = x_1^3 x_2^5 x_3^{10} x_4^{12} x_5^{16}$, $a_{80} = x_1^3 x_2^5 x_3^{10} x_4^{20} x_5^8$.

For $d \geq 5$, $a_{79} = x_1^3 x_2^5 x_3^{10} x_4^{2^{d+1}-12} x_5^{2^d-8}$, $a_{80} = x_1^3 x_2^5 x_3^{2^d-6} x_4^4 x_5^{2^{d+1}-8}$.

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