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Email: tcdhsg@sgu.edu.vn ; Website: http://sj.sgu.edu.vn/

THE ACTION OF THE PRIMITIVE MILNOR OPERATIONS ON THE DICKSON INVARIANTS

Tác động của các toán tử Milnor nguyên thủy trên các bất biến Dickson

Nguyễn Sum

Trường Đại học Sài Gòn

ABSTRACT

In this paper, we present an explicit formula for the action of the primitive Milnor operations on generators of algebra of invariants of the general linear group $GL_n = GL(n, \mathbb{F}_p)$ in the polynomial algebra $P_n = \mathbb{F}_p[x_1, x_2, \ldots, x_n]$ with p a prime number.

Keywords: *Polynomial algebra, cohomology operations, modular invariants.*

TÓM TẮT

Trong bài báo này, chúng tôi trình bày một công thức tường minh về tác động của các toán tử Milnor nguyên thủy trên các phần tử sinh của đại số bất biến của nhóm tuyến tính tổng quát $GL_n = GL(n, \mathbb{F}_p)$ của đại số đa thức $P_n = \mathbb{F}_p[x_1, x_2, \ldots, x_n]$ với p một số nguyên tố lẻ.

Từ khóa: *Đại số đa thức, toán tử đối đồng điều, các bất biến modular.*

1. Introduction

Let p be a prime number. Denote by $GL_n = GL(n, \mathbb{F}_p)$ the general linear group over the prime field $\mathbb{F}_p$ of p elements. This group acts on the polynomial $P_n = \mathbb{F}_p[x_1, x_2, \ldots, x_n]$ in the usual manner. We grade P_n by assigning $\dim x_j = 1$ for $p = 2$ and $\dim x_j = 2$ for $p > 2$. Dickson showed in [2] that the invariant algebra $P_n^{GL_n}$ is a polynomial algebra generated by invariants $Q_{n,s}$, $0 \leqslant s < n$, which are called the Dickson invariants.

Let $\mathcal{A}(p)$ be the mod p Steenrod algebra and denote by $St^R \in \mathcal{A}(p)$ the Milnor operation of type R, where R is a finite sequence of non-negative integers (see Milnor [4], Mùi [5, 6]). For $R = (k)$, $St^{(k)}$ is the Steenrod operation P^k. For $\Delta_i = (0, \ldots, 0, 1)$ of length i, St^{Δ_i} is the primitive Milnor operation in $\mathcal{A}(p)$. This operation was denoted by Q^i in Adams and Wilkerson [1].

Corresponding author: nguyensum@sgu.edu.vn

The Steenrod algebra $\mathcal{A}(p)$ acts on P_n by means of the Cartan formula together with the relations $\beta(x_j) = 0$ and

$$P^k(x_j) = \begin{cases} x_j, & \text{if } k = 0, \\ x_j^p, & \text{if } k = 1, \\ 0, & \text{otherwise,} \end{cases}$$

for $j = 1, 2, \ldots, n$ (see Steenrod and Epstein [8]). Note that P^k is the Steenrod square Sq^k for $p = 2$, and β is the Bockstein operation for $p > 2$. Since this action commutes with the one of GL_n, it induces an inherited action of $\mathcal{A}(p)$ on $P_n^{GL_n}$.

The action of the Milnor operations on the modular invariants of linear groups has partially been studied by Smith and Switzer [7], Wilkerson [14] and the present author [9, 10, 11, 12, 13].

The purpose of the paper is to present a new formula for the action of the primitive Milnor operations St^{Δ_i} on the Dickson invariants.

2. Main Result

First of all, we introduce some notations. Let $(e_1, \ldots, e_n)$ be a sequence of non-negative integers. Following Dickson [2], we define

$$[e_1, e_2, \ldots, e_n] = \begin{vmatrix} x_1^{p^{e_1}} & x_2^{p^{e_1}} & \cdots & x_n^{p^{e_1}} \\ x_1^{p^{e_2}} & x_2^{p^{e_2}} & \cdots & x_n^{p^{e_2}} \\ \vdots & \vdots & \cdots & \vdots \\ x_1^{p^{e_n}} & x_2^{p^{e_n}} & \cdots & x_n^{p^{e_n}} \end{vmatrix}.$$

Denote $L_{n,s} = [0, 1, \ldots, \hat{s}, \ldots, n]$, $0 \leqslant s \leqslant n$, $L_n = L_{n,n} = [0, 1, \ldots, n-1]$. Each $[e_1, e_2, \ldots, e_n]$ is divisible by L_n and $[e_1, e_2, \ldots, e_n]/L_n$ is an GL_n-invariant. Then, Dickson invariants $Q_{n,s}$ are defined by

$$Q_{n,s} = L_{n,s}/L_n, \ 0 \leqslant s < n.$$

By convention, $Q_{n,s} = 0$ for $s < 0$. Note that $Q_{n,0} = L_n^{p-1}$.

Theorem 2.1 (See Dickson [2]). $P_n^{GL_n} = \mathbb{F}_p[Q_{n,0}, Q_{n,1}, \ldots, Q_{n,n-1}]$.

The main result of the paper is the following.

Theorem 2.2. *For any $0 \leqslant s < n$ and $i \geqslant 1$, we have*

$$St^{\Delta_i}(Q_{n,s}) = (-1)^n Q_{n,0}(P_{n,i,s}^p + R_{n,i}^p Q_{n,s}), \tag{2.1}$$

where $P_{n,i,0} = 0$, $P_{n,i,s} = -[0, \ldots, \widehat{s-1}, \ldots, n-1, i-1]/L_n$, for $s > 0$ and

$$R_{n,i} = [0, 1, \ldots, n-2, i-1]/L_n.$$

Note that $R_{n,i} = -P_{n,i,n}$. The case $p = 2$ of (2.1) is also proved by Hưng [3]. It is used to explicitly compute the mod 2 Margolis homology of the Dickson algebra for any i.

We need the following results for the proof of the theorem.

Proposition 2.3 (See Smith-Switzer [7], Wilkerson [14]). *For any* $0 \leqslant s < n$ *and* $1 \leqslant i \leqslant n$, *we have*

$$St^{\Delta_i}(Q_{n,s}) = \begin{cases} (-1)^{s-1}Q_{n,0}, & i = s > 0, \\ (-1)^n Q_{n,0}Q_{n,s}, & i = n, \\ 0, & \textit{otherwise}. \end{cases}$$

The following proposition has been proved in [10, Prop. 1.2] for $p = 2$ and in [11, Prop. 1.2] for $p > 2$.

Proposition 2.4. *For any sequence* $(e_1, \dots, e_n)$ *of non-negative integers, we have*

$$[e_1, \dots, e_{n-1}, e_n + n] = \sum_{s=0}^{n-1}(-1)^{n+s-1}[e_1, \dots, e_{n-1}, e_n + s]Q_{n,s}^{p^{e_n}}.$$

Below is an extension of Proposition 2.3.

Theorem 2.5. *For any* $0 \leqslant s < n$ *and* $i \geqslant 1$, *we have*

$$St^{\Delta_i}(Q_{n,s}) = (-1)^n[0, 1, \dots, \hat{s}, \dots, n-1, i]L_n^{p-2}.$$

Proof. The theorem has been proved in [13, Thm 3.1] for $p > 2$. To make the paper self-contained, we give here a proof of it for p an arbitrary prime.

Since $L_{n,s} = L_nQ_{n,s}$ and St^{Δ_i} is a derivation, we have

$$St^{\Delta_i}(L_{n,s}) = L_nSt^{\Delta_i}(Q_{n,s}) + Q_{n,s}St^{\Delta_i}(L_n). \tag{2.2}$$

From [10, Thm 1.1] for $p = 2$ and [11, Thm 1.1] for $p > 2$, we obtain

$$St^{\Delta_i}(L_{n,s}) = \begin{cases} [i, 1, 2, \dots, \hat{s}, \dots, n], & s > 0, \\ 0, & s = 0. \end{cases}$$

In particular, $St^{\Delta_i}(L_n) = [i, 1, 2, \dots, n-1]$.

If $s = 0$, then $St^{\Delta_i}(L_{n,s}) = 0$ and

$$\begin{aligned} St^{\Delta_i}(L_n) &= [i, 1, 2, \dots, n-1] \\ &= (-1)^{n-1}[1, 2, \dots, n-1, i]. \end{aligned}$$

Combining (2.2), the above equalities and the relation $Q_{n,0} = L_n^{p-1}$, we have

$$\begin{aligned} St^{\Delta_i}(Q_{n,0}) &= -St^{\Delta_i}(L_n)Q_{n,0}/L_n \\ &= (-1)^n[1, 2, \dots, n-1, i]Q_{n,0}/L_n \\ &= (-1)^n[1, 2, \dots, n-1, i]L_n^{p-2}. \end{aligned}$$

Hence, the theorem holds.

If $s > 0$, then $St^{\Delta_i}(L_{n,s}) = [i, 1, 2, \dots, \hat{s}, \dots, n]$. So, using Proposition 2.4, we get

$$\begin{aligned} St^{\Delta_i}(L_{n,s}) &= \sum_{t=0}^{n-1}(-1)^{n-1+t}[i, 1, 2, \dots, \hat{s}, \dots, n-1, t]Q_{n,t} \\ &= (-1)^{n-1}[i, 1, 2, \dots, \hat{s}, \dots, n-1, 0]Q_{n,0} \\ &\quad + (-1)^{n-1+s}[i, 1, 2, \dots, \hat{s}, \dots, n-1, s]Q_{n,s} \\ &= [i, 1, 2, \dots, n-1]Q_{n,s} - [i, 0, 1, \dots, \hat{s}, \dots, n-1]Q_{n,0}. \end{aligned}$$

Combining (2.2), the above equalities and the relation $Q_{n,0} = L_n^{p-1}$, we obtain

$$\begin{aligned} St^{\Delta_i}(Q_{n,s}) &= (St^{\Delta_i}(L_{n,s}) - Q_{n,s}St^{\Delta_i}(L_n))/L_n \\ &= -[i, 0, 1, 2, \dots, \hat{s}, \dots, n-1]Q_{n,0}/L_n \\ &= (-1)^n[0, 1, 2, \dots, \hat{s}, \dots, n-1, i]L_n^{p-2}. \end{aligned}$$

This completes the proof of the theorem. □

We now prove Theorem 2.2.

Proof of Theorem 2.2. By Theorem 2.5, we have

$$\begin{aligned} St^{\Delta_i}(Q_{n,0}) &= (-1)^n[1, 2, \dots, n-1, i]L_n^{p-2} \\ &= (-1)^n\Big([0, 1, \dots, n-2, i-1]/L_n\Big)^p L_n^{2p-2} \\ &= (-1)^n R_{n,i}^p Q_{n,0}^2. \end{aligned}$$

Hence, the theorem is true for $s = 0$.

Assume that $s > 0$. We prove the theorem by induction on i. By Proposition 2.3, the theorem is true for $1 \leqslant i \leqslant n$. Suppose that $i \geqslant n$ and the theorem holds for $1, 2, \dots, i$. Using Proposition 2.4, Theorem 2.5 and the inductive hypothesis, we get

$$\begin{aligned} St^{\Delta_{i+1}}(Q_{n,s}) &= (-1)^n[0, 1, \dots, \hat{s}, \dots, n-1, i+1]L_n^{p-2} \\ &= \sum_{t=0}^{n-1}(-1)^{t-1}[0, \dots, \hat{s}, \dots, n-1, i-n+1+t]Q_{n,t}^{p^{i-n+1}} L_n^{p-2} \\ &= \sum_{t=0}^{n-1}(-1)^{n+t-1}St^{\Delta_{i-n+1+t}}(Q_{n,s})Q_{n,t}^{p^{i-n+1}} \\ &= \sum_{t=0}^{n-1}(-1)^{t-1}Q_{n,0}(P_{n,i-n+1+t,s}^p + R_{n,i-n+1+t}^p Q_{n,s})Q_{n,t}^{p^{i-n+1}} \\ &= (-1)^n Q_{n,0}\Big(\sum_{t=0}^{n-1}(-1)^{n+t-1}P_{n,i-n+1+t,s}^p Q_{n,t}^{p^{i-n+1}} \\ &\qquad + \Big(\sum_{t=0}^{n-1}(-1)^{n+t-1}R_{n,i-n+1+t}^p Q_{n,t}^{p^{i-n+1}}\Big)Q_{n,s}\Big). \end{aligned}$$

Using Proposition 2.4, we have

$$\begin{aligned}
&\sum_{t=0}^{n-1}(-1)^{n+t-1}P^p_{n,i-n+1+t,s}Q^{p^{i-n+1}}_{n,t}\\
&\quad=\sum_{t=0}^{n-1}(-1)^{n+t-1}\Big(-[0,\ldots,\widehat{s-1},\ldots,n-1,i-n+t]/L_n\Big)^pQ^{p^{i-n+1}}_{n,t}\\
&\quad=\Big(\Big(-\sum_{t=0}^{n-1}(-1)^{n+t-1}[0,\ldots,\widehat{s-1},\ldots,n-1,i-n+t]Q^{p^{i-n}}_{n,t}\Big)/L_n\Big)^p\\
&\quad=\Big(-[0,\ldots,\widehat{s-1},\ldots,n-1,i]/L_n\Big)^p=P^p_{n,i+1,s}.
\end{aligned}$$

By a similar computation using Proposition 2.4, we obtain

$$\begin{aligned}
&\sum_{t=0}^{n-1}(-1)^{n+t-1}R^p_{n,i-n+1+t}Q^{p^{i-n+1}}_{n,t}\\
&\quad=\sum_{t=0}^{n-1}(-1)^{n+t-1}\Big([0,1,\ldots,n-2,i-n+t]/L_n\Big)^pQ^{p^{i-n+1}}_{n,t}\\
&\quad=\Big(\Big(\sum_{t=0}^{n-1}(-1)^{n+t-1}[0,1,\ldots,n-2,i-n+t]Q^{p^{i-n}}_{n,t}\Big)/L_n\Big)^p\\
&\quad=\Big([0,1,\ldots,n-2,i]/L_n\Big)^p=R^p_{n,i+1}.
\end{aligned}$$

Thus, the theorem is true for $i+1$. So, the proof is completed. □

Using Theorem 2.2 and Proposition 2.4, we can explicitly compute the action of St^{Δ_i} on the Dickson invariants $Q_{n,s}$ for $i>n$ by explicitly computing $P_{n,i,s}$ and $R_{n,i}$. The cases $i=n+1,\ n+2$ have been computed in [13] by using Theorem 2.5.

Corollary 2.6 (See [13]). *For $0\leqslant s<n$, we have*

$$\begin{aligned}
&St^{\Delta_{n+1}}(Q_{n,s})=(-1)^nQ_{n,0}(-Q^p_{n,s-1}+Q^p_{n,n-1}Q_{n,s}),\\
&St^{\Delta_{n+2}}(Q_{n,s})=(-1)^nQ_{n,0}\Big(Q^{p^2}_{n,s-2}-Q^p_{n,s-1}Q^{p^2}_{n,n-1}+(Q^{p^2+p}_{n,n-1}-Q^{p^2}_{n,n-2})Q_{n,s}\Big).
\end{aligned}$$

By a direct calculation using Proposition 2.4, we easily obtain the following.

Corollary 2.7. *For $0\leqslant s<n$, we have*

$$St^{\Delta_{n+3}}(Q_{n,s})=(-1)^nQ_{n,0}(P^p_{n,n+3,s}+R^p_{n,n+3}Q_{n,s}),$$

where

$$\begin{aligned}
P_{n,n+3,s}&=-Q^{p^2}_{n,s-3}+Q^p_{n,s-2}Q^{p^2}_{n,n-1}+Q_{n,s-1}Q^{p^2}_{n,n-2}-Q_{n,s-1}Q^{p^2+p}_{n,n-1},\\
R_{n,n+3}&=Q^{p^2}_{n,n-3}-Q^{p^2}_{n,n-2}Q_{n,n-1}-Q^p_{n,n-2}Q^{p^2}_{n,n-1}+Q^{p^2+p+1}_{n,n-1}.
\end{aligned}$$

Corollary 2.8. *For any $0\leqslant s<n$ and $i\geqslant 1$, we have*

$$St^{\Delta_i}(Q^{p-1}_{n,0}Q_{n,s})=(-1)^nQ^p_{n,0}P^p_{n,i,s}\in\operatorname{Ker}(St^{\Delta_i}).$$

Proof. Since St^{Δ_i} is a derivation, using Theorem 2.2, we get

$$\begin{aligned} St^{\Delta_i}(Q_{n,0}^{p-1}Q_{n,s}) &= (p-1)Q_{n,0}^{p-2}St^{\Delta_i}(Q_{n,0})Q_{n,s} + Q_{n,0}^{p-1}St^{\Delta_i}(Q_{n,s}) \\ &= (-1)^n\Big(-Q_{n,0}^pR_{n,i}^pQ_{n,s} + Q_{n,0}^p(P_{n,i,s}^p + R_{n,i}^pQ_{n,s})\Big) \\ &= (-1)^nQ_{n,0}^pP_{n,i,s}^p. \end{aligned}$$

The corollary is proved. □

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